Introduction to Proofs - This title is a lie

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Learning Objectives (for this video)

By the end of this video, participants should be able to:

1. Read these learning objectives.
2. State a deep mathematical result that relies on Russell's Paradox.
3. Not accomplish the third learning objective.

Motivation

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This is the motivation for self-reference and Cantor's theorem. That proof, and this motivation, use self-reference.

Many other motivation blocks and theorems in math and computer science use self-reference.

1. Barber's paradox

https://en.wikipedia.org/wiki/Barber_paradox

Example 1 - Barber's paradox

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Does the barber shave himself?

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This is a contradiction or paradox.

2. Self-referential words

https://en.wikipedia.org/wiki/Autological_word

Example 2 - Autological words

A word is autological if it describes itself. e.g. "noun" is a noun, "polysyllabic" (having more than one syllable) has more than one syllable.

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3. Set of all sets

https://en.wikipedia.org/wiki/Russell%27s_paradox

Example 3 - Russell's Paradox

Let R be the collection of all sets that do not contain themselves as elements. In set-builder notation:

$$R = \{x : x \notin x\}$$

For example, $\{1, 2, 3\} \notin \{1, 2, 3\}$, so $\{1, 2, 3\} \in R$.
Is $R \in R$?

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1. If no, then it satisfies the property of R, so $R \in R$.
2. If yes, then it satisfies the property of R, so $R \notin R$.

This is a contradiction or paradox. (We conclude that there is no "set of all sets".)

4. The halting problem

https://en.wikipedia.org/wiki/Halting_problem

Example 4 - Halting Problem

Is there a Python program that can detect whether any code will halt (eventually finish running) have an infinite loop?

5. Gödel shaves himself

https://en.wikipedia.org/wiki/G%C3%B6del%27s_incompleteness_theorems

Example 5 - Godel's Incompleteness Theorem

Does every true statement in math have a proof?

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Example 5 - Godel's Incompleteness Theorem

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Gödel: No, there are true statements that have no proof.

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Further reading

- "Gödel-Escher-Bach: an eternal golden braid" by Douglas Hofstadter. (1979)
- "Gödel's Proof" by Ernest Nagel and James R. Newman. (1958)
- "Logicomix: An Epic Search for Truth" by Apostolos Doxiadis and Christos Papadimitriou (2008).

Reflection

- In what ways are the self-reference examples similar?
- Make your own self-reference paradox.
- Don't reflect on this point; there is nothing to learn here.