Version 1

THE UNIVERSITY OF TORONTO MISSISSAUGA

Fall 2019 FINAL EXAMINATION

MAT102H5F - Introduction to Proofs

Instructors: M. Tvalavadze, S. Fuchs, N. Askaripour, X. Wang, A. Burazin M. Pawliuk

Duration: 3 hours

Aids: None

Last/Surname: ______________________

First/Given Name: ______________________

Student Number: ______________________

Signature: ______________________

This exam contains 20 pages (including this cover page). Check to see if any pages are missing and ensure that all required information at the top of this page has been filled in. You are required to show your work on problems QB, QC, QD, QE, QF, and QG. **Your answers to QA (Multiple Choice) must be entered on Page 20.**

Total Marks = 70

[Use the space below for rough work. This page will ***not*** *be marked, unless you clearly indicate the part of your work that you want us to mark.]*

Question A. [40 MARKS]

Each question in this section is worth two marks for a total of 40 marks.

Each question in this section has only one correct answer.

Your answers must be transferred to the table provided on the final page to be evaluated.
Not doing so will result in a zero grade for this question.

Part 1 [2 MARKS]

Suppose that $p(x)$ is a quadratic polynomial, and that $s(x) = p(x+2)$, for all $x \in \mathbb{R}$.

A. $p(x)$ and $s(x)$ have the same discriminant.

B. The discriminant of $p(x)$ is 2 more than the discriminant of $s(x)$.

C. The discriminant of $s(x)$ is 2 more than the discriminant of $p(x)$.

D. The discriminant of $s(x)$ and the discriminant of $p(x)$ are different, but by more than 2.

E. None of A., B., C., or D.

Part 2 [2 MARKS]

Is the following statement true or false?

$$(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})\,[y \le x \Rightarrow 0 \le x]$$

A. True, because we can take $y = -1$.

B. True, because we can take $y = x$.

C. True, because we can take $y = x^2$.

D. True, because we can take $x = 1$.

E. None of A., B., C., or D.

Part 3 [2 marks]

Which of the following statements is logically equivalent to $P \Leftrightarrow Q$?

A. $P \wedge Q$

B. $P \vee Q$

C. $(P \Rightarrow Q) \wedge (\neg P \Rightarrow \neg Q)$

D. $(P \Rightarrow Q) \wedge (\neg Q \Rightarrow \neg P)$

E. None of A., B., C., or D.

Part 4 [2 marks]

Mike conjectures that for all integers a, b: "If ab is a prime, then a is prime or b is prime." Which of the following strategies can be used in a proof of this statement?

A. Assume that a and b are both not prime. Prove that ab is prime.

B. Assume that ab is prime, and prove that both a and b are not prime.

C. Assume that ab is prime. Derive a contradiction.

D. Assume that ab is prime and that both a and b are not prime. Derive a contradiction.

E. None of A., B., C., or D.

Part 5 [2 marks]

Which of the following sets is not empty?

A. $\emptyset$

B. $\{x : x \in \{\emptyset\}\}$

C. $\mathbb{N} \times \emptyset$

D. $\mathbb{N} \setminus \mathbb{Z}$

E. None of A., B., C., or D.

Part 6 [2 MARKS]

Consider the function $f : \{0,1\} \to \mathbb{R}$ defined by $f(x) = x^2$. Consider the function $g : \{0,1\} \to \mathbb{R}$ defined by $g(x) = x$.

Are these functions equal?

A. No, because they have different codomains.

B. No, because they have different ranges.

C. No, because they have different definitions; $x \neq x^2$.

D. Yes.

E. None of A., B., C., or D.

Part 7 [2 MARKS]

Consider the relation $\leq$ on $\mathbb{R}$.

A. $\leq$ is not reflexive.

B. $\leq$ is not symmetric.

C. $\leq$ is not transitive.

D. $\leq$ is an equivalence relation.

E. None of A., B., C., or D.

Part 8 [2 MARKS]

What is $2019^2 + 2019^4 + 2019^6 + 2019^8$ congruent to modulo 4?

A. 0

B. 1

C. 2

D. 3

E. None of A., B., C., or D.

Part 9 [2 MARKS]

Let $f : A \to B$ be a function, and let $C \subseteq A$. Consider the conjecture:

$$f(A \setminus C) = f(A) \setminus f(C).$$

Is this conjecture true or false?

A. True, because if $A = C$ then the conjecture becomes $\emptyset = \emptyset$ which is true.

B. True, by showing that each set is a subset of the other.

C. False. For example $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ and $C = \{0\}$.

D. False. For example $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ and $C = [0, \infty)$.

E. None of A., B., C., or D.

Part 10 [2 MARKS]

Suppose that $-2 < x < 1$ and $-3 < y < 4$. Which of the following is true?

A. $-7 < |2x + y| < 6$

B. $0 < |2x + y| < 6$

C. $0 \leq |2x + y| < 6$

D. $0 \leq |2x + y| \leq 7$

E. None of A., B., C., or D.

Part 11 [2 MARKS]

What is the range (or image) of the function $f : [-1, 1] \to \mathbb{R}$ defined by $f(x) = \dfrac{1 + x^2}{1 + 2x^2}$?

A. $\left[0, \frac{2}{3}\right]$

B. $\left\{\frac{2}{3}\right\}$

C. $\left[-\frac{2}{3}, \frac{2}{3}\right]$

D. $\left[\frac{2}{3}, 1\right]$

E. None of A., B., C., or D.

Part 12 [2 MARKS]

$$\sum_{k=1}^{2019}(-1)^k k =$$

A. $\frac{2018}{2} - 2019$

B. $\frac{2018}{2} + 2019$

C. $2019 - \frac{2018}{2}$

D. $1 + \frac{2018}{2}$

E. None of A., B., C., or D.

Part 13 [2 MARKS]

Suppose that $f : \mathbb{N} \to \{0, 1\}$ is a function, and Yoshio knows that:

1. $f(2) = 1$,
2. $f(3) = 1$, and
3. $\forall n \in \mathbb{N}\,[f(n) = 1 \Rightarrow f(n+2) = 1]$.

What is the set of all n such that Yoshio knows that $f(n) = 1$?

A. All $n \in \mathbb{N}$ with $n \geq 3$.

B. All $n \in \mathbb{N}$ with $n \geq 2$.

C. All $n \in \mathbb{N}$.

D. All even natural numbers.

E. None of A., B., C., or D.

Part 14 [2 MARKS]

Define a sequence by $a_1 = 1$, $a_2 = 2$ and $a_{n+2} = a_{n+1} \cdot a_n$ for $n \in \mathbb{N}$. What is a_{10}?

A. 2^{10}

B. 2^{34}

C. 2^{21}

D. 2^9

E. None of A., B., C., or D.

Part 15 [2 MARKS]

Let $f : A \to B$ and $g : B \to C$ be functions. Suppose that $g \circ f$ is a surjection. Which of the following must be true?

A. g is a surjection.

B. f is a surjection.

C. g is a bijection.

D. $f \circ g$ is a surjection.

E. None of A., B., C., or D.

Part 16 [2 MARKS]

Which of the following is NOT equivalent to $f : A \to B$ is an injection?

A. $\forall x_1, x_2 \in A$, if $x_1 = x_2$, then $f(x_1) = f(x_2)$.

B. $\forall x_1, x_2 \in A$, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

C. $\forall x_1, x_2 \in A$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

D. For all $b \in B$ there is at most one $a \in A$ such that $f(a) = b$.

E. None of A., B., C., or D.

Part 17 [2 MARKS]

Let A, B be sets. Which of the following is NOT equivalent to $|A| \leq |B|$?

A. There is an injection $f : A \to B$.

B. There is a set $C \subseteq B$ and a injection $g : A \to C$.

C. There is a set $C \subseteq B$ and a bijection $h : A \to C$.

D. $A \subseteq B$.

E. None of A., B., C., or D.

Part 18 [2 MARKS]

Which of the following sets is infinite and countable? (Here, $\mathcal{P}(A)$ is the power set of A.)

A. $\mathcal{P}(\mathbb{N})$

B. $\mathbb{Q} \times \mathbb{Z}$

C. $\mathcal{P}(\{1, 2, 3, \ldots, 2019\})$

D. $\mathcal{P}(\mathbb{R})$

E. None of A., B., C., or D.

Part 19 [2 MARKS]

How many divisors does $1 \cdot 2 \cdot 3 \cdot 4$ have?

A. 4

B. 6

C. 8

D. 24

E. None of A., B., C., or D.

Part 20 [2 MARKS]

Compute $\gcd(2^1 3^2 5^3 7^4, 2^4 3^3 5^2 7^1)$.

A. $2 \cdot 3 \cdot 5 \cdot 7$

B. $2^5 \cdot 3^5 \cdot 5^5 \cdot 7^5$

C. $2^4 \cdot 3^3 \cdot 5^3 \cdot 7^4$

D. $2^1 \cdot 3^2 \cdot 5^2 \cdot 7^1$

E. None of A., B., C., or D.

Long Answer

The following six questions are worth five marks each for a total of 30 marks.

Show your work. Unsupported solutions will receive little or no credit.

Question B. [5 MARKS]

Part 1 [3 MARKS]

Let A, B be sets. Prove that if $A \cap B = \emptyset$, then $(A \times B) \cap (B \times A) = \emptyset$.

Part 2 [2 MARKS]

Is the converse of Part 1 true? If it is then give a proof. If it isn't then provide a counterexample.

Question C. [5 marks]

Let $x, y > 0$ be positive real numbers. Prove that

$$\frac{1}{x} + \frac{1}{y} \geq \frac{4}{x+y}.$$

Question D. [5 MARKS]

Part 1 [2 MARKS]

Negate the statement:

$$(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})\,[y \leq x \Rightarrow 0 \leq x]\,.$$

Part 2 [3 MARKS]

Prove that $\sqrt{6}$ is irrational.

Question E. [5 MARKS]

Part 1 [3 MARKS]

Let E_1, E_2 be equivalence relations on $\mathbb{N}$. Prove that $|E_1| = |E_2|$.

Part 2 [2 MARKS]

George conjectures that "For all sets A, if E_1, E_2 are equivalence relations on A, then $|E_1| = |E_2|$." Show that his conjecture is false by providing a counterexample. (In this part, if something is an equivalence relation, you may state it without proof.)

Question F. [5 MARKS]

Part 1 [3 MARKS]

Apply the Euclidean algorithm to $a = 21$ and $b = 13$ to find $\gcd(13, 21)$.

Part 2 [2 MARKS]

Find integers x, y such that $21x + 13y = 1$.

Question G. [5 MARKS]

Define a sequence recursively as follows: $a_0 = 3, a_n = n + a_{n-1}$ for all $n \in \mathbb{N}$.

Show that for all $n \in \mathbb{N} \cup \{0\}$ we have that $a_n = \dfrac{n^2 + n + 6}{2}$.

*[Use the space below for rough work. This page will **not** be marked, unless you clearly indicate the part of your work that you want us to mark.]*

*[Use the space below for rough work. This page will **not** be marked, unless you clearly indicate the part of your work that you want us to mark.]*

*[Use the space below for rough work. This page will **not** be marked, unless you clearly indicate the part of your work that you want us to mark.]*

*[Use the space below for rough work. This page will **not** be marked, unless you clearly indicate the part of your work that you want us to mark.]*

Total Marks = 70

End of Exam