Version 1

THE UNIVERSITY OF TORONTO MISSISSAUGA

Fall 2019 FINAL EXAMINATION

MAT102H5F - Introduction to Proofs

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Duration: 3 hours

Aids: None

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Please note, once this exam has begun, you **CANNOT** re-write it.

This exam contains 20 pages (including this cover page). Check to see if any pages are missing and ensure that all required information at the top of this page has been filled in. You are required to show your work on problems QB, QC, QD, QE, QF, and QG. **Your answers to QA (Multiple Choice) must be entered on Page 20.**

Total Marks = 70

*[Use the space below for rough work. This page will **not** be marked, unless you clearly indicate the part of your work that you want us to mark.]*

Question A. [40 MARKS]

Each question in this section is worth two marks for a total of 40 marks.

Each question in this section has only one correct answer.

Your answers must be transferred to the table provided on the final page to be evaluated.
Not doing so will result in a zero grade for this question.

Part 1 [2 MARKS]

Suppose that $p(x)$ is a quadratic polynomial, and that $s(x) = p(x+2)$, for all $x \in \mathbb{R}$.

A. $p(x)$ and $s(x)$ have the same discriminant.

B. The discriminant of $p(x)$ is 2 more than the discriminant of $s(x)$.

C. The discriminant of $s(x)$ is 2 more than the discriminant of $p(x)$.

D. The discriminant of $s(x)$ and the discriminant of $p(x)$ are different, but by more than 2.

E. None of A., B., C., or D.

Solution. A. Assume $p(x) = ax^2 + bx + c$ compute the discriminant of $p(x)$: $b^2 - 4ac$ and for $s(x) = a(x+2)^2 + b(x+2) + c = ax^2 + (4a+b)x + 4a + 2b + c$ which is

$$(4a+b)^2 - 4(a)(4a+2b+c) = 16a^2 + 8ab + b^2 - (16a^2 + 8ab + 4ac) = b^2 - 4ac$$

So the discriminants are the same.

Part 2 [2 MARKS]

Is the following statement true or false?

$$(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})\,[y \leq x \Rightarrow 0 \leq x]$$

A. True, because we can take $y = -1$.

B. True, because we can take $y = x$.

C. True, because we can take $y = x^2$.

D. True, because we can take $x = 1$.

E. None of A., B., C., or D.

Solution. A. If $-1 \leq x$, then definitely $0 \leq x$.

Part 3 [2 MARKS]

Which of the following statements is logically equivalent to $P \Leftrightarrow Q$?

A. $P \wedge Q$

B. $P \vee Q$

C. $(P \Rightarrow Q) \wedge (\neg P \Rightarrow \neg Q)$

D. $(P \Rightarrow Q) \wedge (\neg Q \Rightarrow \neg P)$

E. None of A., B., C., or D.

Solution. C. We know this is logically equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$, and the contrapositive of $Q \Rightarrow P$ is $\neg P \Rightarrow \neg Q$, which we know is logically equivalent to $Q \Rightarrow P$.

Part 4 [2 MARKS]

Mike conjectures that for all integers a, b: "If ab is a prime, then a is prime or b is prime."

Which of the following strategies can be used in a proof of this statement?

A. Assume that a and b are both not prime. Prove that ab is prime.

B. Assume that ab is prime, and prove that both a and b are not prime.

C. Assume that ab is prime. Derive a contradiction.

D. Assume that ab is prime and that both a and b are not prime. Derive a contradiction.

E. None of A., B., C., or D.

Solution. D. The original statement is is of the form "$P \implies (Q \vee R)$.", and the proof by contradiction is "Assume P and $\neg(Q \vee R)$. Derive a contradiction." and in this case $\neg(Q \vee R)$ is logically equivalent to $(\neg Q) \wedge (\neg R)$.

The contrapositive is not (A), because the contrapositive method would be "Assume $\neg Q$ and $\neg R$, and prove $\neg P$." In (A), the conclusion is P.

Part 5 [2 MARKS]

Which of the following sets is not empty?

A. $\emptyset$

B. $\{x : x \in \{\emptyset\}\}$

C. $\mathbb{N} \times \emptyset$

D. $\mathbb{N} \setminus \mathbb{Z}$

E. None of A., B., C., or D.

Solution. B. This set contains exactly one element, the empty set.

Part 6 [2 MARKS]

Consider the function $f : \{0, 1\} \to \mathbb{R}$ defined by $f(x) = x^2$. Consider the function $g : \{0, 1\} \to \mathbb{R}$ defined by $g(x) = x$.

Are these functions equal?

A. No, because they have different codomains.

B. No, because they have different ranges.

C. No, because they have different definitions; $x \neq x^2$.

D. Yes.

E. None of A., B., C., or D.

Solution. D. Yes. These two functions have the same domain, same codomain, and $f(0) = g(0)$ and $f(1) = g(1)$, so they must be the same function.

Part 7 [2 MARKS]

Consider the relation $\leq$ on $\mathbb{R}$.

A. $\leq$ is not reflexive.

B. $\leq$ is not symmetric.

C. $\leq$ is not transitive.

D. $\leq$ is an equivalence relation.

E. None of A., B., C., or D.

Solution. B. It is not symmetric, because, for example, $1 \leq 2$, but it is not true that $2 \leq 1$.

Part 8 [2 MARKS]

What is $2019^2 + 2019^4 + 2019^6 + 2019^8$ congruent to modulo 4?

A. 0

B. 1

C. 2

D. 3

E. None of A., B., C., or D.

Solution. A. Note that $2019 \cong 3 \cong -1 \mod 4$. So then

$$2019^{2n} \cong (-1)^{2n} \cong 1 \mod 4$$

for all $n \in \mathbb{N}$. So then

$$2019^2 + 2019^4 + 2019^6 + 2019^8 \cong 1 + 1 + 1 + 1 \cong 0 \mod 4$$

Part 9 [2 MARKS]

Let $f : A \to B$ be a function, and let $C \subseteq A$. Consider the conjecture:

$$f(A \setminus C) = f(A) \setminus f(C).$$

Is this conjecture true or false?

A. True, because if $A = C$ then the conjecture becomes $\emptyset = \emptyset$ which is true.

B. True, by showing that each set is a subset of the other.

C. False. For example $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ and $C = \{0\}$.

D. False. For example $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ and $C = [0, \infty)$.

E. None of A., B., C., or D.

Solution. B. This is one way to show a function/set identity.
In (A), this only shows one special case, not for all sets A, C. C and D are not counterexamples.

Part 10 [2 MARKS]

Suppose that $-2 < x < 1$ and $-3 < y < 4$. Which of the following is true?

A. $-7 < |2x + y| < 6$

B. $0 < |2x + y| < 6$

C. $0 \leq |2x + y| < 6$

D. $0 \leq |2x + y| \leq 7$

E. None of A., B., C., or D.

Solution. D. Note that $2x + y$ is at its largest when $y = 4, x = 1$, which gives $2x + y = 6$. And $2x + y$ is at its smallest when $x = -2, y = -3$ so $2x + y = -7$, and so $|2x + y| = 7$.

Part 11 [2 MARKS]

What is the range (or image) of the function $f : [-1, 1] \to \mathbb{R}$ defined by $f(x) = \dfrac{1 + x^2}{1 + 2x^2}$?

A. $\left[0, \frac{2}{3}\right]$

B. $\left\{\frac{2}{3}\right\}$

C. $\left[-\frac{2}{3}, \frac{2}{3}\right]$

D. $\left[\frac{2}{3}, 1\right]$

E. None of A., B., C., or D.

Solution. A.

Part 12 [2 MARKS]

$$\sum_{k=1}^{2019} (-1)^k k =$$

A. $\frac{2018}{2} - 2019$

B. $\frac{2018}{2} + 2019$

C. $2019 - \frac{2018}{2}$

D. $1 + \frac{2018}{2}$

E. None of A., B., C., or D.

Solution. A. Every pair of consecutive numbers (odd, then even) contributes a total of 1. So you get $\dfrac{2018}{2}$, but then -2019.

Part 13 [2 MARKS]

Suppose that $f : \mathbb{N} \to \{0, 1\}$ is a function, and Yoshio knows that:

1. $f(2) = 1$,
2. $f(3) = 1$, and
3. $\forall n \in \mathbb{N} \, [f(n) = 1 \Rightarrow f(n + 2) = 1]$.

What is the set of all n such that Yoshio knows that $f(n) = 1$?

A. All $n \in \mathbb{N}$ with $n \geq 3$.

B. All $n \in \mathbb{N}$ with $n \geq 2$.

C. All $n \in \mathbb{N}$.

D. All even natural numbers.

E. None of A., B., C., or D.

Solution. B. Assumption 1 and 3 tells you that all the even numbers output 1. Parts 2 and 3 tell you that all odd numbers (at least 3) output 1. So together you know $f(n) = 1$ for all $n \geq 2$.

Part 14 [2 MARKS]

Define a sequence by $a_1 = 1$, $a_2 = 2$ and $a_{n+2} = a_{n+1} \cdot a_n$ for $n \in \mathbb{N}$. What is a_{10}?

A. 2^{10}

B. 2^{34}

C. 2^{21}

D. 2^{9}

E. None of A., B., C., or D.

Solution. B. Note that $a_n = 2^{F_n}$ where F_n is the nth Fibonacci number. i.e.

$$a_3 = 2^1, a_4 = 2^1 \cdot 2^1 = 2^{1+1}, a_5 = 2^2 \cdot 2^1 = 2^{2+1}, a_6 = 2^3 \cdot 2^2 = 2^5.$$

So by computing it out we see $a_{10} = 2^{34}$.

Part 15 [2 MARKS]

Let $f : A \to B$ and $g : B \to C$ be functions. Suppose that $g \circ f$ is a surjection. Which of the following must be true?

A. g is a surjection.

B. f is a surjection.

C. g is a bijection.

D. $f \circ g$ is a surjection.

E. None of A., B., C., or D.

Solution. A.

Part 16 [2 MARKS]

Which of the following is NOT equivalent to $f : A \to B$ is an injection?

A. $\forall x_1, x_2 \in A$, if $x_1 = x_2$, then $f(x_1) = f(x_2)$.

B. $\forall x_1, x_2 \in A$, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

C. $\forall x_1, x_2 \in A$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

D. For all $b \in B$ there is at most one $a \in A$ such that $f(a) = b$.

E. None of A., B., C., or D.

Solution. A. Since this is the definition of f being a function.

Part 17 [2 MARKS]

Let A, B be sets. Which of the following is NOT equivalent to $|A| \leq |B|$?

A. There is an injection $f : A \to B$.

B. There is a set $C \subseteq B$ and a injection $g : A \to C$.

C. There is a set $C \subseteq B$ and a bijection $h : A \to C$.

D. $A \subseteq B$.

E. None of A., B., C., or D.

Solution. D. It's true that $A \subseteq B \implies$ —A— $\leq |B|$, but the converse is not true. e.g. $|\{4,5\}| = 2 \leq 3 = |\{6,7,8\}|$, but $\{4,5\} \not\subseteq \{6,7,8\}$.

Part 18 [2 MARKS]

Which of the following sets is infinite and countable? (Here, $\mathcal{P}(A)$ is the power set of A.)

A. $\mathcal{P}(\mathbb{N})$

B. $\mathbb{Q} \times \mathbb{Z}$

C. $\mathcal{P}(\{1, 2, 3, \ldots, 2019\})$

D. $\mathcal{P}(\mathbb{R})$

E. None of A., B., C., or D.

Solution. B. This is countable since both $\mathbb{Q}$ and $\mathbb{N}$ are countable. Note that (A) is uncountable by Cantor's theorem, (C) is finite, and (D) is uncountable by Cantor's theorem.

Part 19 [2 MARKS]

How many divisors does $1 \cdot 2 \cdot 3 \cdot 4$ have?

A. 4

B. 6

C. 8

D. 24

E. None of A., B., C., or D.

Solution. C. The divisors are $1, 2, 3, 4, 6, 8, 12, 24$. By definition, a divisor is a positive integer.

Part 20 [2 MARKS]

Compute $\gcd(2^1 3^2 5^3 7^4, 2^4 3^3 5^2 7^1)$.

A. $2 \cdot 3 \cdot 5 \cdot 7$

B. $2^5 \cdot 3^5 \cdot 5^5 \cdot 7^5$

C. $2^4 \cdot 3^3 \cdot 5^3 \cdot 7^4$

D. $2^1 \cdot 3^2 \cdot 5^2 \cdot 7^1$

E. None of A., B., C., or D.

Solution. D. Take the minimum of each prime power.

Long Answer

The following six questions are worth five marks each for a total of 30 marks.

Show your work. Unsupported solutions will receive little or no credit.

Question B. [5 MARKS]

Part 1 [3 MARKS]

Let A, B be sets. Prove that if $A \cap B = \emptyset$, then $(A \times B) \cap (B \times A) = \emptyset$.

Solution. We will prove this by contradiction. Assume that $(A \times B) \cap (B \times A) \neq \emptyset$. So there is an $(x, y) \in (A \times B) \cap (B \times A)$.

$$\begin{aligned}
&(x, y) \in A \times B \wedge (x, y) \in B \times A \\
&\Rightarrow x \in A \wedge y \in B \wedge x \in B \wedge y \in A \\
&\Rightarrow x \in A \wedge x \in B \\
&\Rightarrow x \in A \cap B \\
&\Rightarrow A \cap B \neq \emptyset
\end{aligned}$$

As desired.

Part 2 [2 MARKS]

Is the converse of Part 1 true? If it is then give a proof. If it isn't then provide a counterexample.

Solution. The converse is true. We again prove it by contrapositive. Assume $A \cap B \neq \emptyset$. Let $x \in A \cap B$. So $x \in A$ and $x \in B$. Then $(x, x) \in A \times B$ and $(x, x) \in B \times A$. So $(x, x) \in (A \times B) \cap (B \times A)$. Thus $(A \times B) \cap (B \times A) \neq \emptyset$.

Question C. [5 MARKS]

Let $x, y > 0$ be positive real numbers. Prove that

$$\frac{1}{x} + \frac{1}{y} \geq \frac{4}{x+y}.$$

Solution. Let $x, y > 0$. Note $x + y > 0$ and $xy > 0$. Note

$$\begin{aligned}
&(x-y)^2 \geq 0 \\
&\Rightarrow x^2 - 2xy + y^2 \geq 0 \\
&\Rightarrow x^2 + 2xy + y^2 \geq 4xy \\
&\Rightarrow (x+y)^2 \geq 4xy \\
&\Rightarrow x + y \geq \frac{4xy}{x+y} \qquad \text{since } x + y > 0 \\
&\Rightarrow \frac{x+y}{xy} \geq \frac{4}{x+y} \qquad \text{since } xy > 0 \\
&\Rightarrow \frac{1}{x} + \frac{1}{y} \geq \frac{4}{x+y}
\end{aligned}$$

As desired.

Question D. [5 MARKS]

Part 1 [2 MARKS]

Negate the statement:

$$(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})\,[y \leq x \Rightarrow 0 \leq x]\,.$$

Solution. We will negate this one step at a time:

$$\begin{aligned}
&\neg(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})\,[y \leq x \Rightarrow 0 \leq x] \\
&= (\exists x \in \mathbb{R})\neg(\exists y \in \mathbb{R})\,[y \leq x \Rightarrow 0 \leq x] \\
&= (\exists x \in \mathbb{R})(\forall y \in \mathbb{R})\neg\,[y \leq x \Rightarrow 0 \leq x] \\
&= (\exists x \in \mathbb{R})(\forall y \in \mathbb{R})\,[y \leq x \wedge \neg(0 \leq x)] \qquad [\text{ since } \neg(P \implies Q) = P \wedge \neg Q] \\
&= (\exists x \in \mathbb{R})(\forall y \in \mathbb{R})\,[y \leq x \wedge 0 > x]
\end{aligned}$$

Part 2 [3 MARKS]

Prove that $\sqrt{6}$ is irrational.

Solution. Suppose for the sake of contradiction that $\sqrt{6}$ is rational. So it can be written as $\sqrt{6} = \dfrac{a}{b}$ where $a, b \in \mathbb{N}$ and it is written in lowest terms (i.e. we've cancelled all common factors). (Note that $\sqrt{6}$ is positive, so clearly we can assume a, b are both positive.)

Now

$$\begin{aligned}
\sqrt{6} &= \frac{a}{b} \\
\implies 6 &= \frac{a^2}{b^2} \\
\implies b^2 6 &= a^2 \\
\implies b^2 \cdot 2 \cdot 3 &= a^2
\end{aligned}$$

Since 2 is a prime and $2|a^2$, we must have $2|a$ by Euclid's lemma. So then there is an integer k such that $a = 2k$. So

$$\begin{aligned}
b^2 \cdot 2 \cdot 3 &= a^2 = (2k)^2 = 4k^2 \\
\implies b^2 \cdot 3 &= 2k^2
\end{aligned}$$

Since $2|(3b^2)$, and $2 \not| 3$, by Euclid's lemma we must have $2|b^2$, and so $2|b$.

Thus we see that 2 is a common factor of both a and b, a contradiction.

Question E. [5 MARKS]

Part 1 [3 MARKS]

Let E_1, E_2 be equivalence relations on $\mathbb{N}$. Prove that $|E_1| = |E_2|$.

Solution. We will prove that every such equivalence relation E on $\mathbb{N}$ is countable, so then we must have $|E_1| = |E_2|$.

$\boxed{|\mathbb{N}| \leq |E|}$ Since E is an equivalence relation on $\mathbb{N}$, E is reflexive, and so $(n, n) \in E$ for all $n \in \mathbb{N}$. This guarantees that $f : \mathbb{N} \to E$ given by $f(n) = (n, n)$ is an injection. So by definition, $|\mathbb{N}| \leq |E|$.

$\boxed{|E| \leq |\mathbb{N}|}$ Since E is a relation on $\mathbb{N}$, we know that E is a collection of pairs of natural numbers. In other words, $E \subseteq \mathbb{N} \times \mathbb{N}$. By a lemma from class (or using the identity function), this tells us that $|E| \leq |\mathbb{N} \times \mathbb{N}|$. By a result from class, we know that $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$. So we have proved that $|E| \leq |\mathbb{N}|$.

Together, by the Cantor-Schroeder-Bernstein theorem, we have $|E| = |\mathbb{N}|$.

Part 2 [2 MARKS]

George conjectures that "For all sets A, if E_1, E_2 are equivalence relations on A, then $|E_1| = |E_2|$." Show that his conjecture is false by providing a counterexample. (In this part, if something is an equivalence relation, you may state it without proof.)

Solution.
For example, let $A = \{0, 1\}$, and let $E_1 = \{(0, 0), (1, 1)\}$ and $E_2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. Clearly these are both equivalence relations on A, but $|E_1| = 2 \neq 4 = |E_2|$.

Question F. [5 MARKS]

Part 1 [3 MARKS]

Apply the Euclidean algorithm to $a = 21$ and $b = 13$ to find $\gcd(13, 21)$.

Solution. Note

$$\begin{aligned} 21 &= 1 \cdot 13 + 8 \\ 13 &= 1 \cdot 8 + 5 \\ 8 &= 1 \cdot 5 + 3 \\ 5 &= 1 \cdot 3 + 2 \\ 3 &= 1 \cdot 2 + \boxed{1} \\ 2 &= 2 \cdot 1 \end{aligned}$$

So $\gcd(13, 21) = 1$.

Part 2 [2 MARKS]

Find integers x, y such that $21x + 13y = 1$.

Solution. Note

$$\begin{aligned} 1 &= 3 - 2 \\ &= (8 - 5) - (5 - 3) \\ &= 8 - 2(5) + 3 \\ &= (21 - 13) - 2(13 - 8) + (8 - 5) \\ &= 21 - 3(13) + 3(8) - 5 \\ &= 21 - 3(13) + 3(21 - 13) - (13 - 8) \\ &= 4(21) - 7(13) + 8 \\ &= 4(21) - 7(13) + (21 - 13) \\ &= 5(21) - 8(13) \end{aligned}$$

So $x = 5$ and $y = 13$ satisfies the equation.

Question G. [5 MARKS]

Define a sequence recursively as follows: $a_0 = 3, a_n = n + a_{n-1}$ for all $n \in \mathbb{N}$.

Show that for all $n \in \mathbb{N} \cup \{0\}$ we have that $a_n = \dfrac{n^2 + n + 6}{2}$.

Solution. Let $P(n)$ be the statement "$a_n = \dfrac{n^2+n+6}{2}$". We will prove $\forall n \in \mathbb{N}, P(n)$ by induction.

Base. Note $a_1 = 1 + a_0 = 1 + 3 = 4$ and $\dfrac{1^2+1+6}{2} = \frac{8}{2} = 4$. So $P(1)$.

Induction. Assume $P(n)$ for a particular $n \in \mathbb{N}$.

Note that

$$\begin{aligned} a_{n+1} &= (n+1) + a_n && \text{by definition} \\ &= (n+1) + \frac{n^2+n+6}{2} && \text{by the IH} \\ &= \frac{2(n+1) + n^2 + n + 6}{2} \\ &= \frac{n^2+3n+8}{2} \\ &= \frac{(n^2+2n+1) + (n+1) + 6}{2} \\ &= \frac{(n+1)^2 + (n+1) + 6}{2} \end{aligned}$$

So $P(n+1)$.

Grading.

- 1pt: Stated the $P(n)$ explicitly and correctly.
- 1pt: Proved the base case explicitly and correctly.
  - Do not award this point if the student starts with the conclusion, and then derives a true statement.
- 1pt: The induction hypothesis was explicitly assumed for a particular $n \in \mathbb{N}$,
- 1pt: The use of the IH was pointed out (correctly).
- 1pt: The structure of the proof of the inductive step was correct.
  - Do not award this point if the student starts with the conclusion, and then derives a true statement.
  - Award this point only if the mathematical idea of the inductive step is mostly correct.

[Use the space below for rough work. This page will ***not*** *be marked, unless you clearly indicate the part of your work that you want us to mark.]*

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Total Marks = 70

End of Exam