THE UNIVERSITY OF TORONTO MISSISSAUGA

AUGUST 2020 FINAL EXAMINATION

MAT102Y5S - Introduction to Proofs

Instructors: M. Pawliuk, Q. Wang

Date: August 21, 2020

Time: 1:00pm - 4:00pm (EDT)

Duration: 3 hours

Aids: Course Textbook, Course Notes.

Submission

- **You must upload your completed Final Exam on Crowdmark by 4:00pm (EDT) Friday August 21, 2020**.
- Late submissions will not be accepted.
- If you require additional space, please insert extra pages.
- You are not required to print out this exam. You may submit clear photos or scans of your work. You may also submit screenshots of your work (from, for example, a tablet).

Permitted Resources

During the final exam:

1. You may use any resources (course notes, textbook, videos) that have been posted to Quercus by instructors or TAs.
2. You may use personal notes related to official course material (from reading the textbook, participating in lectures/tutorials, completing problem sets).
3. Do not use personal notes related to other material (e.g. notes created by studying external websites)
4. Do not communicate with anyone other than the instructors.
5. Do not use Piazza except to make private posts to instructors.
6. Do not use any online resources other than Quercus and Crowdmark.

Academic Integrity

By submitting this final exam you affirm that your submission represents entirely your own efforts. You confirm that:

- You have not copied any portion of this work.
- You have not allowed someone else in the course to copy this work.
- You understand the consequences of violating the University's academic integrity policies as outlined in the *Code of Behaviour on Academic Matters*.

Long Answer

The following 12 questions are worth five marks each for a total of 60 marks.

Show your work. Unsupported solutions will receive little or no credit.

Question A. [5 MARKS]

1. A remote controlled robot starts at the integer 0.
2. On day n (where $n \in \mathbb{N}$) you can give the robot exactly one instruction: either $L(n)$ to move left n units, or $R(n)$ to move right n units.

For example, one possible set of instructions is: $L(1), L(2), R(3), L(4), \ldots$ and it will reach the integers $0, -1, -3, 0, -4, \ldots$.

Prove that the robot can reach every integer. (Here "to reach an integer" means to begin or end the day at that integer.)

Question B. [5 MARKS]

Consider the following two mathematical statements (where x, y are real numbers):

$$P : (\forall x > 0)(\exists y > 0)(x^2 - y^2 > 0)$$
$$Q : (\exists x > 0)(\forall y > 0)(x^2 - y^2 > 0)$$

Part 1 [2 MARKS]

Prove or disprove P.

Part 2 [2 MARKS]

Prove or disprove Q.

Part 3 [1 MARK]

Write down the negation of P (without using $\neg$).

Question C. [5 MARKS]

For all parts of this question your final answer should not include "¬".

Let $P(x)$ be: "If x is not a multiple of 3, then $x + 1$ is a multiple of 3 or $x + 2$ is a multiple of 3."

Part 1 [1 MARK]

Write the contrapositive of $P(x)$.

Part 2 [1 MARK]

Write the converse of $P(x)$.

Part 3 [1 MARK]

Write the negation of $P(x)$.

Part 4 [2 MARKS]

Is $P(6)$ true or false? Explain.

Question D. [5 MARKS]

Suppose A, B, M are three sets satisfying

$$A \cap M = B \cap M = A \cap B \text{ and}$$
$$A \cup B \cup M = A \cup B$$

Part 1 [2 MARKS]

Prove that $M \subseteq A \cup B$.

Part 2 [3 MARKS]

Prove that $M = A \cap B$.

Question E. [5 MARKS]

Consider the relation $S = \{(2,2),(3,3),(4,4),(1,2),(2,4),(4,2),(4,1),(3,2),(4,3)\}$ on the set $X = \{1,2,3,4\}$.

Part 1 [1 MARK]

Is S reflexive? Justify your answer.

Part 2 [1 MARK]

Is S symmetric? Justify your answer.

Part 3 [1 MARK]

Is S transitive? Justify your answer.

Part 4 [2 MARKS]

A relation R on a set X is defined to be centred if

$$(\exists x \in X)(\forall y \in X)[(x,y) \in R \text{ or } (y,x) \in R].$$

Is S centred? Justify your answer.

Question F. [5 MARKS]

Find the largest value of c such that $(x-1)^2 + 1 \geq cx$ for all real numbers x. (Do not use calculus.)

Question G. [5 MARKS]

For $n \in \mathbb{N}$, let $S_n = \sum_{i=0}^{5n-1} 2^i$.

Part 1 [1 MARK]

Calculate S_1, and write out the sum S_2 without sigma notation.

Part 2 [4 MARKS]

Prove that $\forall n \in \mathbb{N}, \quad 31 | S_n$.

Question H. [5 MARKS]

Let $a_1 = 1$, and let $a_{k+1} = a_1 + a_2 + \ldots + a_k$, where $k \in \mathbb{N}$.

Let c be a real number.

Let $b_1 = c$, and let $b_{k+1} = b_1 + b_2 + \ldots + b_k$, where $k \in \mathbb{N}$.

Part 1 [2 MARKS]

Compute a_{2020}. (You do not need to justify your answer).

Part 2 [3 MARKS]

Prove that for every $k \in \mathbb{N}$ that $b_k = ca_k$.

Question I. [5 MARKS]

Part 1 [2 MARKS]

Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ by $f(a, b) = (-1)^a(b - 1)$. Prove that f is surjective.

Part 2 [3 MARKS]

Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} x & x \leq 0 \\ 1/x & x > 0 \end{cases}$$

Prove that g is injective.

Question J. [5 MARKS]

Let $A = \{2^n : n \in \mathbb{N}\}$ and let $B = \{3^n : n \in \mathbb{N}\}$.

Part 1 [2 MARKS]

Prove that A and B are disjoint.

Part 2 [3 MARKS]

Prove that $A \cup B$ is countable.

Question K. [5 MARKS]

Recall that $\mathcal{P}(X)$ is the power set of X.

Suppose that $f : A \to B$ is a bijection.

Prove that $|\mathcal{P}(A)| = |\mathcal{P}(B)|$.

Question L. [5 MARKS]

Let $a, b \in \mathbb{N}$. Let $\gcd(a, b)$ be the greatest common divisor of a and b.

Part 1 [2 MARKS]

Professor Sr. Proofini incorrectly claims that $\gcd(a, b) = \gcd(ka, b)$ for all $k, a, b \in \mathbb{N}$. Find a counterexample.

Part 2 [3 MARKS]

Jr. Proofini correctly claims that $\gcd(ka, kb) = k \cdot \gcd(a, b)$ for all $k, a, b \in \mathbb{N}$. Prove that she is correct.

Total Marks = 60