MAT102H5 Y - SUMMER 2020 - PROBLEM SET 5 - SOLUTIONS

Submission

- **You must submit your completed problem set on Crowdmark by 5:00 pm (EDT) Friday July 24, 2020**.
- Late assignments will not be accepted.
- Consider submitting your assignment well before the deadline.
- If you require additional space, please insert extra pages.
- You do not need to print out this assignment; you may submit clear pictures/scans of your work on lined paper, or screenshots of your work.
- **You must include a signed and completed version of this cover page**.

Additional Instructions

You must justify and support your solution to each question.

Academic Integrity

You are encouraged to work with your fellow students while working on questions from the problem sets. However, the writing of your assignment must be done without any assistance whatsoever.

By signing this statement I affirm that this assignment represents entirely my own efforts. I confirm that:

- I have not copied any portion of this work.
- I have not allowed someone else in the course to copy this work.
- This is the final version of my assignment and not a draft.
- I understand the consequences of violating the University's academic integrity policies as outlined in the *Code of Behaviour on Academic Matters.*

By signing this form I agree that the statements above are true.

If I do not agree with the statements above, I will not submit my assignment and will consult my course instructor immediately.

Student Name: ____

Student Number: ____

Signature: ____

Date: ____

Problem 1. Consider the function $f(x) = \dfrac{x+3}{x+1}$ and the numbers defined by

$$a_1 = 1, \quad a_{n+1} = f(a_n), \quad b_n = |a_n - \sqrt{3}|, \quad S_n = \sum_{i=1}^{n} b_i$$

(1) What is the range of $f(x)$ if the domain is $x \neq -1$?
(2) What is the range of $f(x)$ if the domain is $[\sqrt{3}, \infty)$?
(3) What is the range of $f(x)$ if the domain is $[0, \sqrt{3}]$?
(4) Prove that for every natural number n, $b_n \leq \dfrac{(\sqrt{3}-1)^n}{2^{n-1}}$ (hint: Use mathematical induction if you prefer.)
(5) Prove that for every natural number n, $S_n < \dfrac{2\sqrt{3}}{3}$

Solution.

(1) we have that

$$f(x) = \frac{x+3}{x+1} = 1 + \frac{2}{x+1}$$

Clearly $\frac{2}{x+1} \neq 0$. On the other hand, $\forall y \neq 1$, let $x = \frac{2}{y-1} - 1$ is well-defined, and $f(x) = y$. We conclude that the range is $(-\infty, 1) \cup (1, \infty)$.

(2) $f(x)$ is decreasing for $x > -1$, as a result if the domain is $[\sqrt{3}, \infty)$, the range will be $(1, \sqrt{3}]$.

(3) $f(x)$ is decreasing for $x > -1$, as a result if the domain is $[0, \sqrt{3}]$, the range will be $[\sqrt{3}, 3]$.

(4) We prove the statement by mathematical induction.

- For $n = 1$, $b_1 = \sqrt{3} - 1$, the statement is true.
- Suppose now that the statement is true for a particular natural number k , i.e. $b_k \leq \frac{(\sqrt{3}-1)^k}{2^{k-1}}$. We discuss two situations:

  (a) if $a_k < \sqrt{3}$, we have that $\sqrt{3} - a_k = b_k =\leq \frac{(\sqrt{3}-1)^k}{2^{k-1}}$ thus $a_k \geq \sqrt{3} - \frac{(\sqrt{3}-1)^k}{2^{k-1}}$

  $$a_{k+1} = \frac{a_k+3}{a_k+1} > \sqrt{3}$$

  This implies that

  $$b_{k+1} = a_{k+1} - \sqrt{3} = \frac{a_k+3}{a_k+1} - \sqrt{3} \leq \frac{\sqrt{3} - \frac{(\sqrt{3}-1)^k}{2^{k-1}} + 3}{\sqrt{3} - \frac{(\sqrt{3}-1)^k}{2^{k-1}} + 1} - \sqrt{3} = \frac{(\sqrt{3}-1)^{k+1}}{2^{k-1}(\sqrt{3} - \frac{(\sqrt{3}-1)^k}{2^{k-1}} + 1)} \leq \frac{(\sqrt{3}-1)^{k+1}}{2^k}$$

  (b) if $a_k > \sqrt{3}$, we have that $a_k - \sqrt{3} = b_k =\leq \frac{(\sqrt{3}-1)^k}{2^{k-1}}$ thus $a_k \leq \sqrt{3} + \frac{(\sqrt{3}-1)^k}{2^{k-1}}$

  $$a_{k+1} = \frac{a_k+3}{a_k+1} < \sqrt{3}$$

  This implies that

  $$b_{k+1} = \sqrt{3} - a_{k+1} = \sqrt{3} - \frac{a_k+3}{a_k+1} < \sqrt{3} - \frac{\sqrt{3} + \frac{(\sqrt{3}-1)^k}{2^{k-1}} + 3}{\sqrt{3} + \frac{(\sqrt{3}-1)^k}{2^{k-1}} + 1} \leq \frac{(\sqrt{3}-1)^{k+1}}{2^k}$$

  We conclude that in both cases $b_{k+1} \leq \frac{(\sqrt{3}-1)^{k+1}}{2^k}$.
- By mathemtical induction, it follows that $\forall n \in \mathbb{N}$, $b_n \leq \dfrac{(\sqrt{3}-1)^n}{2^{n-1}}$. The statement is thus proved.

(5) By previous result,

$$S_n = \sum_{k=1}^{n} b_k \leq \sum_{k=1}^{n} \frac{(\sqrt{3}-1)^k}{2^{k-1}} = (\sqrt{3}-1) \sum_{k=1}^{n} \frac{(\sqrt{3}-1)^{k-1}}{2^{k-1}} \leq (\sqrt{3}-1) \frac{1}{1 - \frac{\sqrt{3}-1}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

according to the sum of geometric series.

Grading. This question is worth 5 points.

(1) Pb1.1: 1 point, all or nothing,
(2) **Pb1.2 and Pb 1.3** 1 point in all, all or nothing.
(3) **Pb1.4**: 2 points. 1 point for noticing the alternating sign of $a_n - \sqrt{3}$, 1 point for the proof of statement.
(4) **Pb1.5**: 1 point, all or nothing.

Problem 2. An ice cream shop offers the following deal for repeat customers:

(1) After you buy 10 ice cream cones, you get an 11th one for free.
(2) After that, if you buy 9 ice cream cones, you get a 10th one for free.
(3) After that, if you buy 8 ice cream cones, you get a 9th one for free.
(4) (Similarly for buying $7, 6, 5, 4, 3, 2$, then 1 ice cream cone.)
(5) Finally after all that they give you one free bonus ice cream cone.

You must:

(1) Compute the total discount (i.e. the percentage of ice cream that is free) that a customer receives if they receive all 11 free ice cream cones.
(2) Suppose that instead of starting at "buy 10 get 1 free" it started at "buy n, get one free" (still going down by one each time), where n is a natural number. Compute the total discount the customer will receive if they receive all $n+1$ free ice cream cones.
(3) Compute the discount in part (2) for $n = 2020$.

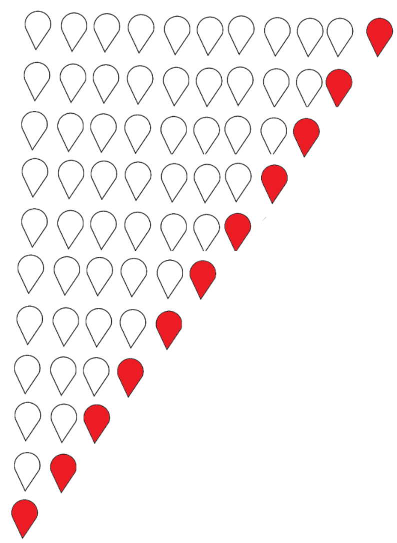

FIGURE 1. All ice creams for $n = 10$. The red ice creams are the free ones.

Solution. We will answer question 2 in general, then answer 1 and 3. Suppose the discount starts at "buy n, get one free".

$$\text{Total bought} = n + (n-1) + \ldots + 3 + 2 + 1 = \frac{n(n+1)}{2} \qquad \text{formula from textbook}$$

$$\text{Total free} = n + 1 \qquad \text{from question}$$

$$\text{Total received} = \text{Total bought} + \text{Total free} = \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2}$$

So the total discount, $D(n)$, is

$$D(n) = \frac{\text{Total free}}{\text{Total received}} = \frac{n+1}{\frac{(n+1)(n+2)}{2}} = \frac{2}{n+2}.$$

So $\boxed{D(10) = \frac{2}{12} \approx 16.67\%}$ and $\boxed{D(2020) = \frac{2}{2022} \approx 0.01\%}$.

Grading. This question is worth 5 points. 1 point each for a correct final answer of (1) and (3), no part marks. Part (2) is worth 3 points, they do not need to simplify their answer.
Please do not spend a lot of time grading this question.

Problem 3. 4.6.29 from the textbook. Let x be a non-zero real number, such that $x + \frac{1}{x}$ is an integer. Prove that for all $n \in \mathbb{N}$, the number $x^n + \frac{1}{x^n}$ is also an integer.

Solution. We will prove this by induction. Let $P(n)$ be the statement $x^n + \frac{1}{x^n}$ is an integer. Note that $P(1)$ is an assumption in the question statement. However, we will explicitly need $P(0)$ so we check that:

$\boxed{P(0)}$. Note that $x^0 + \frac{1}{x^0} = 1 + \frac{1}{1} = 2 \in \mathbb{Z}$. (Note that we do not have $x = 0$, so we don't have to worry about 0^0.)

$\boxed{\text{Inductive hypothesis}}$. Assume $P(n-1)$ and $P(n)$, for a particular $n \in \mathbb{N}$. Note that by $P(1)$ and $P(n)$:

$$\left(x^n + \frac{1}{x^n}\right)\left(x + \frac{1}{x}\right) \in \mathbb{Z}.$$

Note

$$\begin{aligned}\left(x^n + \frac{1}{x^n}\right)\left(x + \frac{1}{x}\right) &= x^{n+1} + \frac{x}{x^n} + \frac{x^n}{x} + \frac{1}{x^{n+1}} \\ &= \left(x^{n+1} + \frac{1}{x^{n+1}}\right) + \left(x^{n-1} + \frac{1}{x^{n-1}}\right)\end{aligned}$$

By $P(n-1)$ the number on the right is an integer. Therefore, by rearranging:

$$x^{n+1} + \frac{1}{x^{n+1}} = \left(x^n + \frac{1}{x^n}\right)\left(x + \frac{1}{x}\right) - \left(x^{n-1} + \frac{1}{x^{n-1}}\right).$$

Since the difference of integers is an integer, we have $P(n+1)$.

Solution. An alternate approach, which avoids proving $P(0)$, but repeats the work of the induction step, is to prove $P(2)$ as follows:

$$\begin{aligned}\left(x + \frac{1}{x}\right)\left(x + \frac{1}{x}\right) &= x^2 + \frac{x}{x} + \frac{x}{x} + \frac{1}{x^2} \\ &= \left(x^2 + \frac{1}{x^2}\right) + (2)\end{aligned}$$

Therefore, by rearranging:

$$x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)\left(x + \frac{1}{x}\right) - 2.$$

Which is a difference of integers.

Grading.

- 1pt: Stated the $P(n)$ explicitly and correctly.
- 1pt: Proved the base cases explicitly and correctly.
  - This requires proving either $P(0)$ or $P(2)$.
  - Do not award this point if the student starts with the conclusion, and then derives a true statement.
- 1pt: The induction hypothesis was explicitly assumed for a particular $n \in \mathbb{N}$, and its use was pointed out (correctly).
  - Use of strong induction is acceptable. But leave a comment.
- 1pt: The structure of the proof of the inductive step was correct.
  - Do not award this point if the student starts with the conclusion, and then derives a true statement.
  - Award this point only if the mathematical idea of the inductive step is mostly correct.

Problem 4. Recall that the Fibonacci numbers are defined by $F_0 = 0, F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \in \mathbb{N} \cup \{0\}$.

(1) Make and prove an (if and only if) conjecture about which Fibonacci numbers are multiples of 3.
(2) Make a conjecture about which Fibonacci numbers are multiples of 2020. (You do not need to prove your conjecture.) How many base cases would a proof by induction of your conjecture require?

The following lemmas will be useful:

Lemma 1. *$\forall x \in \mathbb{Z}$, x is a multiple of 3 if and only if $2x$ is a multiple of 3.*

Lemma 2. *$F_{n+4} = 3F_{n+1} + 2F_n$, for every $n \in \mathbb{N}$.*

Proof. Note

$$\begin{aligned} F_{n+4} &= F_{n+3} + F_{n+2} \\ &= (F_{n+2} + F_{n+1}) + (F_{n+1} + F_n) \\ &= F_{n+2} + 2F_{n+1} + F_n \\ &= (F_{n+1} + F_n) + 2F_{n+1} + F_n \\ &= 3F_{n+1} + 2F_n \end{aligned}$$

□

Theorem 1. *For all $n \in \mathbb{N} \cup \{0\}$, $3|F_n$ if and only if n is a multiple of 4.*

Proof. We will prove this by working in intervals of length 4. Let $P(n)$ be the statement "F_{4n} is a multiple of 3, and $F_{4n+1}, F_{4n+2}, F_{4n+3}$ are not multiples of 3".

Base case For $n = 0$, observe that

- $F_{4(0)} = F_0 = 0 = 3(0)$ which is a multiple of 3,
- $F_{4(0)+1} = F_1 = 1$ which is a not multiple of 3,
- $F_{4(0)+2} = F_2 = 1$ which is a not multiple of 3,
- $F_{4(0)+3} = F_3 = F_2 + F_1 = 1 + 1 = 2$ which is a not multiple of 3.

Induction step Suppose that $P(n)$ is true for a particular $n \in \mathbb{N} \cup \{0\}$. So there is an $m \in \mathbb{Z}$ such that $F_{4n} = 3m$, and $F_{4n+1}, F_{4n+2}, F_{4n+3}$ are not multiples of 3.

Note

$$\begin{aligned} F_{4(n+1)} &= F_{4n+4} \\ &= 3F_{4n+1} + 2F_{4n} && \text{By Q1.1} \\ &= 3F_{4n+1} + 2(3m) && \text{By IH} \\ &= 3(F_{4n+1} + 2m) \end{aligned}$$

Since $F_{4n+1} + 2m \in \mathbb{Z}$, we have shown that $F_{4(n+1)}$ is a multiple of 3.

Now we will show that $F_{4(n+1)+1}, F_{4(n+1)+2}$, and $F_{4(n+1)+3}$ are not multiples of 3. Let ϵ be one of $1, 2, 3$ (since all the arguments are identical).

$$\begin{aligned} F_{4(n+1)+\epsilon} &= F_{4n+4+\epsilon} \\ &= 3F_{4n+1+\epsilon} + 2F_{4n+\epsilon} && \text{By Q1.1} \end{aligned}$$

The number $3F_{4n+1+\epsilon}$ is obviously divisible by 3, and by the IH, $F_{4n+\epsilon}$ is not divisible by 3, so then $2F_{4n+\epsilon}$ is not divisible by 3.

So $P(n + 1)$ is true.

□

Grading. This question is worth 3 points; and we're mostly looking for correct ideas.

- 1 point for the idea of of connecting F_{n+4} and F_n.
- 1 point for a more or less correct proof that all F_{4n} are multiples of 3.
- 1 point for a more or less correct proof that the other numbers are not multiples of 3.

Theorem 2. *For all* $n \in \mathbb{N} \cup \{0\}$, $2020|F_n$ *if and only if* n *is a multiple of* 150.

Solution. This is related to the fact that $2020 = 4 \cdot 5 \cdot 101$, and the following facts:

(1) $4|F_n$ if and only if n is a multiple of 6.
(2) $5|F_n$ if and only if n is a multiple of 5.
(3) $101|F_n$ if and only if n is a multiple of 50.

It's related to the fact that 150 is the least common multiple of $6, 5$, and 50.
There are two ways to answer the question "how many base cases"?

(1) 150, because you need to check all the numbers $F_0, \ldots, F_{149}$.
(2) 50, because you only need to check the number $F_0, \ldots, F_{49}$. This uses the related facts cited above.

Grading. This is worth 2 points. 1 point for the correct answer "150", and 1 point for any reasonable answer (be generous).