MAT102H5 Y - SUMMER 2020 - PROBLEM SET 6

Submission

- **You must submit your completed problem set on Crowdmark by 5:00 pm (EDT) Friday August 7, 2020**.
- Late assignments will not be accepted.
- Consider submitting your assignment well before the deadline.
- If you require additional space, please insert extra pages.
- You do not need to print out this assignment; you may submit clear pictures/scans of your work on lined paper, or screenshots of your work.
- **You must include a signed and completed version of this cover page**.

Additional Instructions

You must justify and support your solution to each question.

Academic Integrity

You are encouraged to work with your fellow students while working on questions from the problem sets. However, the writing of your assignment must be done without any assistance whatsoever.

By signing this statement I affirm that this assignment represents entirely my own efforts. I confirm that:

- I have not copied any portion of this work.
- I have not allowed someone else in the course to copy this work.
- This is the final version of my assignment and not a draft.
- I understand the consequences of violating the University's academic integrity policies as outlined in the *Code of Behaviour on Academic Matters.*

By signing this form I agree that the statements above are true.
If I do not agree with the statements above, I will not submit my assignment and will consult my course instructor immediately.

Student Name: ____

Student Number: ____

Signature: ____

Date: ____

Problem 1. Suppose that $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ are functions. Define $h : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ by $h(n) = (f(n), g(n))$. For each of the following, prove that it is true, or provide a counterexample.

(1) If f, g are both injections, then h is an injection.
(2) If f, g are both surjections, then h is a surjection.
(3) If f, g are both bijections, then h is a bijection.

Problem 2. A set A is defined to be "Dedekind infinite" if there exists an injection $f : A \to A$ that is not a surjection.

(1) Prove that $\mathbb{N}$ and $\mathbb{R}$ are Dedekind infinite.
(2) Prove that $\{1\}$ is not Dedekind infinite. (Hint: Your proof should begin with "Let $f : \{1\} \to \{1\}$ be an injection.")
(3) Prove that $\{1, 2, 3\}$ is not Dedekind infinite.

Problem 3 (You do not need to submit a solution to this question.). Challenge! Prove by induction that $\{1, \ldots, 2020\}$ is not Dedekind infinite.

Problem 4. This is a game played by Jr. Proofini and a fox, and it takes place on the set $\mathbb{Z} \times \mathbb{Z}$, which we call "the integer lattice". Jr. Proofini is trying to catch the fox.

- Before the game begins, the fox chooses two secret integer numbers a and b.
- The fox is not allowed to change these numbers during the game.
- Jr. Proofini does not know what a and b are.
- During the night, the fox moves from its current position (x, y) to the position $(x + a, y + b)$, and hides there until the next night.
- During the day, Jr. Proofini chooses any one integer lattice point (x, y) she wants, and searches for the fox.
- If Jr. Proofini searches on the exact grid point that the fox is on, then she has caught the fox!
- The game begins on the first night with the fox moving to (a, b).

Prove that no matter what integers a, b the fox picks to start the game, Jr. Proofini can always catch the fox after finitely many days.
(**Hint**: As a warm-up, first solve the question assuming that $a = 0$ and b is either 1 or -1.)

Problem 5 (You do not need to submit a solution to this question.). Challenge! Show that even if the fox is allowed to start the first night at any lattice point it wants (not just (a, b)), that Jr. Proofini can still find the fox after finitely many days.
Find your own ridiculous adaptation of this problem, where Jr. Proofini can still find the fox. Share your adaptation (but not your solution) with the class on Piazza.