MAT102H5 Y - SUMMER 2020 - PROBLEM SET 6 - SOLUTIONS

Submission

- **You must submit your completed problem set on Crowdmark by 5:00 pm (EDT) Friday August 7, 2020**.
- Late assignments will not be accepted.
- Consider submitting your assignment well before the deadline.
- If you require additional space, please insert extra pages.
- You do not need to print out this assignment; you may submit clear pictures/scans of your work on lined paper, or screenshots of your work.
- **You must include a signed and completed version of this cover page**.

Additional Instructions

You must justify and support your solution to each question.

Academic Integrity

You are encouraged to work with your fellow students while working on questions from the problem sets. However, the writing of your assignment must be done without any assistance whatsoever.

By signing this statement I affirm that this assignment represents entirely my own efforts. I confirm that:

- I have not copied any portion of this work.
- I have not allowed someone else in the course to copy this work.
- This is the final version of my assignment and not a draft.
- I understand the consequences of violating the University's academic integrity policies as outlined in the *Code of Behaviour on Academic Matters.*

By signing this form I agree that the statements above are true.
If I do not agree with the statements above, I will not submit my assignment and will consult my course instructor immediately.

Student Name: ____

Student Number: ____

Signature: ____

Date: ____

Problem 1. Suppose that $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ are functions. Define $h : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ by $h(n) = (f(n), g(n))$. For each of the following, prove that it is true, or provide a counterexample.

(1) If f, g are both injections, then h is an injection.

Solution. This is true, and in fact all that is needed is that one of the two functions is an injection. Suppose that $f : \mathbb{N} \to \mathbb{N}$ is an injection, $g : \mathbb{N} \to \mathbb{N}$ is a function, and h is defined as above.

Suppose that $n, m \in \mathbb{N}$ are such that $h(n) = h(m)$. So then $((f(n), g(n)) = (f(m), g(m))$. Which means that $f(n) = f(m)$ (and $g(n) = g(m)$).

Since f is an injection, $n = m$, as desired.

(2) If f, g are both surjections, then h is a surjection.

Solution. This is false. Suppose that f, g are both the identity function, i.e. $f(n) = n$ and $g(n) = n$ for all $n \in \mathbb{N}$. These are obviously surjections, and in fact are bijections.

Note that

$$\text{range}(h) = \{h(n) : n \in \mathbb{N}\} = \{(f(n), g(n)) : n \in \mathbb{N}\} = \{(n, n) : n \in \mathbb{N}\}.$$

In other words, the range of f only includes pairs whose first and second coordinates are the same. So then, for example, $(1, 2) \in \mathbb{N} \times \mathbb{N}$ but $(1, 2) \notin \text{range}(h)$.

(3) If f, g are both bijections, then h is a bijection.

Solution. This is false for the same reason as Q1.2.

Grading. This question is worth 6 points.

(1) Part 1 is worth 2 points.
(2) Part 2/3 are worth 4 points total: 1 point each for the correct answer (False/False), 1 point for a plausible counterexample to one of them, 1 point for a complete argument for one of them.

Problem 2. A set A is defined to be "Dedekind infinite" if there exists an injection $f : A \to A$ that is not a surjection.

(1) Prove that $\mathbb{N}$ and $\mathbb{R}$ are Dedekind infinite.

Solution. Consider the function $f : \mathbb{N} \to \mathbb{N}$ given by $f(n) = n + 1$. This is clearly an injection ($n + 1 = m + 1 \implies n = m$) and is not a surjection because $1 \notin \text{range}(f)$ (the smallest output of f is the value 2.)

For $\mathbb{R}$, both arctan and exponential functions like e^x work; injectivity is because of their known inverse functions tan and $\ln x$, and neither function achieves $-\pi/2$.

An alternate example similar to the $\mathbb{N}$ example above, is:

$$f(x) = \begin{cases} x & x \leq 0 \\ x + 1 & 0 < x \end{cases}$$

This will omit all values in $(0, 1]$.

(2) Prove that $\{1\}$ is not Dedekind infinite. (Hint: Your proof should begin with "Let $f : \{1\} \to \{1\}$ be an injection.")

Solution. Let $f : \{1\} \to \{1\}$ be an injection. Note that $f(1) \in \{1\}$, so the only choice is that $f(1) = 1$. So f is a surjection.

We have just shown that every injection $f : \{1\} \to \{1\}$ is also a surjection.

(3) Prove that $\{1, 2, 3\}$ is not Dedekind infinite.

Solution. Suppose that $f : \{1, 2, 3\} \to \{1, 2, 3\}$ is an injection. This means that $f(1), f(2)$, and $f(3)$ are different elements of $\{1, 2, 3\}$. The only way this is possible is if $\{f(1), f(2), f(3)\} = \{1, 2, 3\}$ and so f is a surjection.

Solution. An alternate solution is to use the pigeonhole principle from PS2.
Claim: Suppose A is a finite set with at least 2 elements. If $f : A \to A$ is an injection, then f is a surjection.

We will prove this by contrapositive. Suppose that $2 \leq |A| = n$, a natural number, and $f : A \to A$ is not a surjection. This means that $|\text{range}(f)| \leq n - 1$, which is at least 1. By the pigeonhole principle, there are two different $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$. So f is not an injection.

Solution. A third (boss) proof is to prove this without using the pigeonhole principle! This is a subtle and challenging use of induction. This is actually a proof of the pigeonhole principle!

Let $P(n)$ be the statement "$\{1, 2, \ldots, n\}$ is not Dedekind infinte.". Question 2.2 is the base case.

Suppose for the sake of contradiction that $P(n)$ is true, but $P(n+1)$ is false. Let $f : \{1, \ldots, n+1\} \to \{1, \ldots, n+1\}$ be an injection that is not a surjection.

<u>Case 1</u>. Suppose $f(n+1) = n+1$. Note $\exists N \leq n$ with $f(x) \neq N$ for all $x \in \{1, \ldots, n\}$. Then f restricted to $\{1, \ldots, n\}$ is still an injection, and still is not a surjection since it still misses N.

<u>Case 2</u>. Suppose $f(n+1) = N < n+1$.

<u>Case 2.1</u>. If there is no x with $f(x) = n+1$, then define the function $g : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $g(k) = f(k)$ is still an injection, and it is a surjection because the only $f(k) = N$ is $f(n+1)$. So there is no $g(k) = N$.

<u>Case 2.2</u>. If there is an $x \leq n$ with $f(x) = n+1$, then define the function $g : \{1, \ldots, n\} \to \{1, \ldots, n\}$ by $g(k) = f(k)$, unless $k = x$, in which case $g(x) = N$. This is still an injection, and it is not a surjection because it misses the same points in $\{1, \ldots, n\}$ that f missed.

Grading. This question is worth 6 points.

(a) Part 1 is worth 3 points:
- 1 point each for correct functions,
- 1 point if the injective/surjective part is obvious or they provide sufficient justification.

(b) Part 2 is worth 1 point, all or nothing.

(c) Part 2 is worth 2 points.

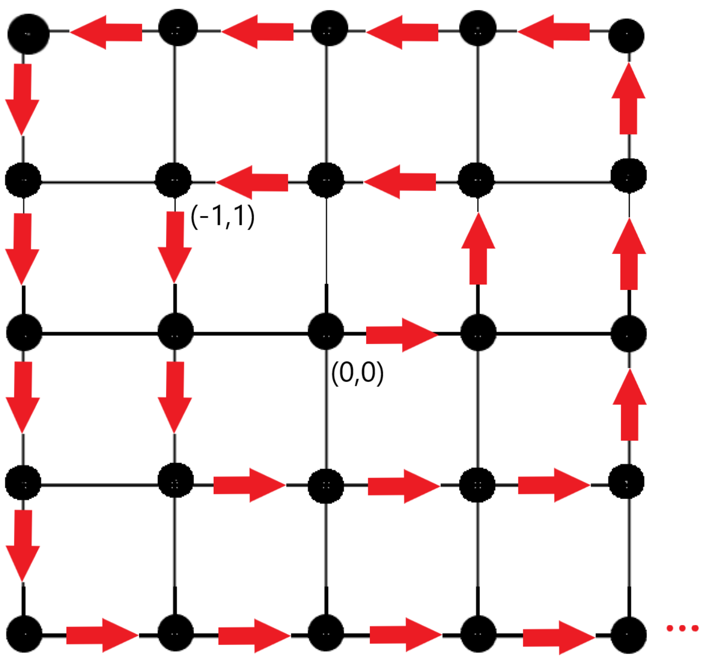
Problem 3 (**You do not need to submit a solution to this question.**). **Challenge!** Prove by induction that $\{1, \ldots, 2020\}$ is not Dedekind infinite.

Problem 4. This is a game played by Jr. Proofini and a fox, and it takes place on the set $\mathbb{Z} \times \mathbb{Z}$, which we call "the integer lattice". Jr. Proofini is trying to catch the fox.

- Before the game begins, the fox chooses two secret integer numbers a and b.
- The fox is not allowed to change these numbers during the game.
- Jr. Proofini does not know what a and b are.
- During the night, the fox moves from its current position (x, y) to the position $(x + a, y + b)$, and hides there until the next night.
- During the day, Jr. Proofini chooses any one integer lattice point (x, y) she wants, and searches for the fox.
- If Jr. Proofini searches on the exact grid point that the fox is on, then she has caught the fox!
- The game begins on the first night with the fox moving to (a, b).

Prove that no matter what integers a, b the fox picks to start the game, Jr. Proofini can always catch the fox after finitely many days.
(**Hint**: As a warm-up, first solve the question assuming that $a = 0$ and b is either 1 or -1.)

Solution. Note that by using a spiral argument as suggested in the diagram, there is an enumeration $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots$ of $\mathbb{Z} \times \mathbb{Z}$ using the index set $\mathbb{N}$.



The big idea is that each day Jr. Proofini can eliminate one possible starting position for the fox. She will eliminate a position, not by looking at that position but by asking herself "If the fox chose, for example, $(-1, 1)$ as its (a, b), then where would it be today?". She will project how much the fox would have moved from that potential starting position and look at that projection.

More precisely, on day $n \in \mathbb{N}$, Jr. Proofini should check the position (nx_n, ny_n). If the fox is not there, then Jr. Proofini knows that the fox did not start at the point $(a, b) = (x_n, y_n)$, because n days later the fox would be at $(na, nb) = (nx_n, ny_n)$.

Since $\mathbb{Z} \times \mathbb{Z} = \{(x_i, y_i) : i \in \mathbb{N}\}$, we know that (a, b) must be one of these (x_i, y_i). So on the ith day (where i is the index of (a, b) in the enumeration of $\mathbb{Z} \times \mathbb{Z}$), Jr. Proofini will find the fox.

For example, in the enumeration in the picture above, if $(a, b) = (-1, 1)$, then on the 5th day Jr proofini will search $(-5, 5)$ and find the fox.

Grading. This question is worth 8 points.

(1) 4 points total for arguing that $\mathbb{Z} \times \mathbb{Z}$ is countable; a formal proof is not necessary, but their argument should be clear.
(2) 2 points for the idea of "(nx, ny)", i.e. eliminate days one by one by looking at where they will be after n days assuming they started at (x, y).
(3) 2 points for a clear proof.

Problem 5 (You do not need to submit a solution to this question.). Challenge! Show that even if the fox is allowed to start the first night at any lattice point it wants (not just (a, b)), that Jr. Proofini can still find the fox after finitely many days.
Find your own ridiculous adaptation of this problem, where Jr. Proofini can still find the fox. Share your adaptation (but not your solution) with the class on Piazza.

Solution. There are many silly adaptations for this; here's mine.
In this version of the game the fox also choose a secret (finite) amount of days $N \in \mathbb{N}$ to wait before it starts playing the game, an Jr. Proofini will not be able to find the fox in those first N days.

However, it can be shown that Jr. Proofini can always find the fox in finitely many days.