MAT102H5 Y - SUMMER 2020 - QUIZ 4 - SOLUTIONS

Submission

- **You must submit your completed Quiz on Crowdmark by 6:00pm (EDT) Tuesday July 28, 2020**.
- Late submissions will not be accepted.
- You should start uploading your quiz no later than 5:45pm.
- If you require additional space, please insert extra pages.
- You do not need to print out this quiz; you may submit clear pictures/scans of your work on lined paper, or screenshots/PDFs of your work.

Additional Instructions

You must justify and support your solution to each question.

Permitted Resources

During the quiz:

(1) You may use any resources (course notes, textbook, videos) that have been posted to Quercus by instructors or TAs.
(2) You may use personal notes related to official course material (from reading the textbook, participating in lectures/tutorials, completing problem sets).
(3) Do not use personal notes related to other material (e.g. notes created by studying external websites)
(4) Do not communicate with anyone other than the instructors.
(5) Do not use Piazza.
(6) Do not use any online resources other than Quercus and Crowdmark.

Academic Integrity

By submitting this quiz you affirm that your submission represents entirely your own efforts. You confirm that:

- You have not copied any portion of this work.
- You have not allowed someone else in the course to copy this work.
- You understand the consequences of violating the University's academic integrity policies as outlined in the *Code of Behaviour on Academic Matters*.

Problem 1 [10 points]

(1) Jr. Proofini wants to write down the sum of squares $1 + 4 + 9 + \ldots + n^2$, and she mistakenly writes:

$$\sum_{i=1}^{n} n^2.$$

Explain her mistake to her, and write down the correct summation notation.

Solution. The mistake is that her square term doesn't depend on the (dummy) index i, so she is adding the end term n^2 repeatedly. A correct summation notation is:

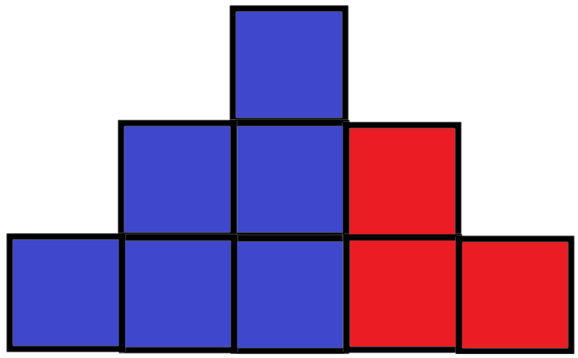
$$\sum_{i=1}^{n} i^2.$$

Grading. This question is worth 4 points: 2 points for correctly pointing out the error, and 2 points for the correct summation.

(2) Let $n \in \mathbb{N}$. Prove that

$$\sum_{i=0}^{n-1}(2i+1) = \left(\sum_{i=0}^{n} i\right) + \left(\sum_{i=0}^{n-1} i\right).$$

Solution. This question was motivated by the following image ($n = 3$). The horizontal rows are the sum of the odds $(1 + 3 + 5)$, but the blue left $(1 + 2 + 3)$ and red right $(1 + 2)$ halves are each the sums of consecutive integers.



Note that this is not a formal proof.

Solution. Note

$$\begin{aligned}
\sum_{i=0}^{n-1}(2i+1) &= \sum_{i=0}^{n-1}(i+i+1) \\
&= \sum_{i=0}^{n-1} i + \sum_{i=0}^{n-1} i + \sum_{i=0}^{n-1} 1 \\
&= \sum_{i=0}^{n-1} i + \left(\sum_{i=0}^{n-1} i\right) + n && \text{Since } \sum_{i=0}^{n-1} 1 = n \\
&= \sum_{i=0}^{n} i + \sum_{i=0}^{n-1} i
\end{aligned}$$

The final equality is because

$$\sum_{i=0}^{n} i = 1 + 2 + \ldots + (n-1) + n = \left(\sum_{i=0}^{n-1} i\right) + n.$$

Grading. This question is worth 6 points.

(a) 2 points for correctly breaking up the sum across $2i + 1$.
(b) 2 points for the correct sum of 1s to be n.
(c) 2 points for absorbing the n into the sum.

Problem 2 [10 points]

Consider the sequence defined recursively as

$$a_1 = 1 \qquad \text{and} \qquad a_{n+1} = n + 1 - a_n, \forall n \in \mathbb{N}.$$

Hint: Induction is not needed for one of these two parts.

(1) Prove that for all $n \in \mathbb{N}$ that $a_{n+2} = 1 + a_n$.

Solution. We solve this question directly, without induction. Let $n \in \mathbb{N}$.
Note

$$\begin{aligned} a_{n+2} &= n + 2 - a_{n+1} && \text{By definition} \\ &= n + 2 - (n + 1 - a_n) && \text{By definition} \\ &= 1 + a_n \end{aligned}$$

Grading. This is worth 5 points: 3 points for the algebra, and 2 points for a clear explanation.

(2) Prove that for all <u>even</u> $n \in \mathbb{N}$ that $a_n = \frac{n}{2}$.

Solution. The first 10 terms are:

$$1, 1, 2, 2, 3, 3, 4, 4, 5, 5$$

Solution. We prove this by induction on the evens. Let $P(n)$ be the statement "$a_n = \frac{n}{2}$".

Base, $n = 2$ Note that $a_2 = 2 - a_1 = 2 - 1 = 1 = \frac{2}{2}$.

Induction Suppose $P(n)$ is true for a particular even natural number n.
Note

$$\begin{aligned} a_{n+2} &= 1 + a_n && \text{By part 1} \\ &= 1 + \frac{n}{2} && \text{By IH} \\ &= \frac{2+n}{2} \end{aligned}$$

As desired.

Grading.

- 1pt: Stated the $P(n)$ explicitly and correctly.
- 1pt: Proved the base case explicitly and correctly.
  - Do not award this point if the student starts with the conclusion, and then derives a true statement.
- 1pt: The induction hypothesis was explicitly assumed for a particular even $n \in \mathbb{N}$, and its use was pointed out (correctly).
  - Use of strong induction is acceptable. But leave a comment.
- 2pt: The structure of the proof of the inductive step was correct.
  - Do not award this point if the student starts with the conclusion, and then derives a true statement.
  - Award this point only if the mathematical idea of the inductive step is mostly correct.