MAT102H5 Y - SUMMER 2020 - QUIZ 5 - SOLUTIONS

Submission

- **You must submit your completed Quiz on Crowdmark by 6:00pm (EDT) Tuesday August 11, 2020**.
- Late submissions will not be accepted.
- You should start uploading your quiz no later than 5:45pm.
- If you require additional space, please insert extra pages.
- You do not need to print out this quiz; you may submit clear pictures/scans of your work on lined paper, or screenshots/PDFs of your work.

Additional Instructions

You must justify and support your solution to each question.

Permitted Resources

During the quiz:

(1) You may use any resources (course notes, textbook, videos) that have been posted to Quercus by instructors or TAs.
(2) You may use personal notes related to official course material (from reading the textbook, participating in lectures/tutorials, completing problem sets).
(3) Do not use personal notes related to other material (e.g. notes created by studying external websites)
(4) Do not communicate with anyone other than the instructors.
(5) Do not use Piazza.
(6) Do not use any online resources other than Quercus and Crowdmark.

Academic Integrity

By submitting this quiz you affirm that your submission represents entirely your own efforts. You confirm that:

- You have not copied any portion of this work.
- You have not allowed someone else in the course to copy this work.
- You understand the consequences of violating the University's academic integrity policies as outlined in the *Code of Behaviour on Academic Matters*.

Problem 1 [10 points]

(1) Show that for all sets A, B, C if $|A| \leq |B|$ and $|B| = |C|$, then $|A| \leq |C|$. You may use any results from class without proving them.

Solution. Suppose that $|A| \leq |B|$ and $|B| = |C|$. By definition, there is an injection $f : A \to B$ and a bijection $g : B \to C$.

Since g is a bijection, it is also an injection. Thus $g \circ f : A \to C$ is an injection, by a result from class. Thus $|A| \leq |C|$ by definition.

Grading. 5 points
- (a) 2 points for using the function definitions.
- (b) 1 point for realizing that a bijection is an injection.
- (c) 2 points for a clear and complete solution.

(2) Let $A, B_1, B_2, ...B_{2020}, C$ be sets such that $|A| \leq |B_1| \leq |B_2| \leq \ldots \leq |B_{2020}| \leq |C|$. If $|C| = |A|$, then

$$|B_1| = |B_2| = \ldots = |B_{2020}|.$$

You may use any results from class without proving them.

Solution. By transitivity of $\leq$ for cardinality, we observe that $|A| \leq |B_i| \leq |A|$ for all $i = 1, 2, \ldots, 2020$.

By the Cantor-Schroeder-Bernstein Theorem, $|A| = |B_i|$ for all $i = 1, 2, \ldots, 2020$.

Therefor $|B_1| = |B_2| = \ldots = |B_{2020}|$, and they are all in fact equal to the cardinality of A.

Grading. 5 points
- (a) 3 points for using CSB
- (b) 2 points for a clear and complete solution.

Problem 2 [10 points]

A function $f : B \to C$ is defined to be a double cover if

$$\forall c \in C, \exists b_1, b_2 \in B, \text{ such that } b_1 \neq b_2 \text{ and } f(b_1) = c \text{ and } f(b_2) = c.$$

(1) Prove that the following function is a double cover, $f : [1, \infty) \to \mathbb{N}$ defined by $f(x)$ is "x rounded down to the nearest integer". For example, $f(2.99) = 2, f(\pi) = 3$, and $f(9) = 9$.

Solution. Let $n \in \mathbb{N}$. Note that $n \neq n + 0.5$, and $f(n) = n$ and $f(n + 0.5) = n$. Thus f is a double cover.

Grading. 4 points for a complete solution.

(2) Prove that if $g : A \to B$ is a surjection and $f : B \to C$ is a double cover, then $f \circ g$ is a double cover.

Solution. Suppose that $g : A \to B$ is a surjection and $f : B \to C$ is a double cover.

Let $c \in C$. Since f is a double cover, there are $b_1, b_2 \in B$, such that $b_1 \neq b_2$ and $f(b_1) = c$ and $f(b_2) = c$.

Since g is a surjection, there is an $a_1 \in A$ such that $g(a_1) = b_1$. Similarly, there is an $a_2 \in A$ such that $g(a_2) = b_2$. Since $b_1 \neq b_2$, and g is a function we must have $a_1 \neq a_2$.

Thus $a_1 \neq a_2$ and $f(g(a_1)) = c$ and $f(g(a_2)) = c$. So $f \circ g$ is a double cover by definition.

Grading. 6 points total

(a) 2 points for starting with a $c \in C$.
(b) 2 points for correctly using the definitions of double cover and surjective on f and g.
(c) 2 points for completeness and clarity.