TUTORIALS FOR MAT102 - SUMMER 2020

This document was originally created by Micheal Pawliuk in Summer 2020 while he was a professor at the University of Toronto Mississauga (UTM).

The materials were originally created for MAT102 Introduction to Proofs, a course a UTM.

Tutorial for Week 2 - TA Version

By the end of this tutorial, students should be able to:

(1) Prove a true statement about divisibility, primes and composites (using definition unwinding).

(2) Create a counterexample to a false statement about divisibility, primes, and composites.

(3) Detect common flaws in definition unwinding proofs.

(4) Make mathematical conjectures from playing.

Overview Thinking mathematically involves playing with mathematical objects, creating conjectures, and then trying to prove that they are true or false. In this tutorial, students will create conjectures about divisibility, primes, and composite numbers.

Suggested Activities

Activity 1. In the first 5 minutes, introduce yourself, and explain how the tutorials will be run (What are **your** expectations? What are **their** expectations?).

The following was (unintentionally) skipped in LEC0101, so this will be the first time some of them are seeing this (although it should be a definition they know from high school).

Activity 2. State the formal definitions for a natural number n to be ...

(1) a prime number.

(2) a composite number.

(3) a divisor of a number m.

For the next problem, allow them to come up with multiple modifications. For these ones it's a matter of strengthening the premise. Part 2 was a quiz problem in Fall 2019. Many students failed to check that their factors were (1) not 1, and (2) not $n^2 - 1$.

Activity 3. For each of the following conjectures, find a counterexample if it is false, and modify the statement to be true. If it is true, then prove it.

(1) If a, b are natural numbers, then ab is a composite number.

(2) If n is a natural number, then $n^2 - 1$ is a composite number.

(3) **Harder**. Every square natural number $n > 1$ has an odd number of divisors.

Activity 4. The following proofs have flaws in them. Find the flaws, explain in words what the issue is, and then correct it.

Theorem 1. *The sum of two odd integers is even.*

Proof. Let x and y be odd integers. Since x is odd, there is an integer n such that $x = 2n + 1$. Since y is odd, there is an integer $y = 2n + 1$. So

$$x + y = 2n + 1 + 2n + 1 = 4n + 2 = 2(2n + 1).$$

Since n is an integer, so is $2n + 1$. So by definition, $x + y$ is even. $\square$

The issue here is that they are using the same dummy variable.

Theorem 2. *Every integer is odd.*

Proof. Let x be an integer. Note that

$$x = 2\left(\frac{x}{2}\right) = 2\left(\frac{x}{2}\right) - 1 + 1 = 2\left(\frac{x}{2} - \frac{1}{2}\right) + 1.$$

Let $k = \frac{x}{2} - \frac{1}{2}$. So then $x = 2k + 1$. □

The issue is that this k need not be an integer.

The next activity is similar to a problem set question. There are many approaches here, and do not rush them to an answer. Let them play around and make conjectures, and try things. The final answer isn't so important, it's more about getting them to play around. (One possible solution is: hop left n, right n+1 is the same as hoping 1 right.)

If they want more of a challenge, modify the jump lengths to consecutive primes,

Activity 5. A frog is going to hop along lilypads. There is one lilypad for ever integer, and it starts at number 0. The frog's first jump is of length 1 (in either direction), then length 2 (in either direction), then length 3, etc.

Show that the frog can reach lilypad 2020.

Tutorial for Week 2 - Handout

By the end of this tutorial, students should be able to:

(1) Prove a true statement about divisibility, primes and composites (using definition unwinding).

(2) Create a counterexample to a false statement about divisibility, primes, and composites.

(3) Detect common flaws in definition unwinding proofs.

(4) Make mathematical conjectures from playing.

Suggested Activities

Activity 1. State the formal definitions for a natural number n to be ...

(1) a prime number. (2) a composite number. (3) a divisor of a number m.

Activity 2. For each of the following conjectures, find a counterexample if it is false, and modify the statement to be true. If it is true, then prove it.

(1) If a, b are natural numbers, then ab is a composite number.

(2) If n is a natural number, then $n^2 - 1$ is a composite number.

(3) **Harder**. Every square natural number $n > 1$ has an odd number of divisors.

Activity 3. The following proofs have flaws in them. Find the flaws, explain in words what the issue is, and then correct it.

Theorem 3. *The sum of two odd integers is even.*

Proof. Let x and y be odd integers. Since x is odd, there is an integer n such that $x = 2n + 1$. Since y is odd, there is an integer $y = 2n + 1$. So

$$x + y = 2n + 1 + 2n + 1 = 4n + 2 = 2(2n + 1).$$

Since n is an integer, so is $2n + 1$. So by definition, $x + y$ is even. □

Theorem 4. *Every integer is odd.*

Proof. Let x be an integer. Note that

$$x = 2\left(\frac{x}{2}\right) = 2\left(\frac{x}{2}\right) - 1 + 1 = 2\left(\frac{x}{2} - \frac{1}{2}\right) + 1.$$

Let $k = \frac{x}{2} - \frac{1}{2}$. So then $x = 2k + 1$. □

Activity 4. A frog is going to hop along lilypads. There is one lilypad for ever integer, and it starts at number 0. The frog's first jump is of length 1 (in either direction), then length 2 (in either direction), then length 3, etc.

Show that the frog can reach lilypad 2020.

Tutorial for Week 3 - TA Version

By the end of this tutorial, students should be able to:

(1) Translate a mathematical statement into English.

(2) Identify when an implication is vacuously true.

(3) Prove a statement involving quantifiers.

Overview The students have a quiz today (Tuesday May 19, 2020 at 5pm EDT). Remind them to check Quercus for details.

In Week 2 we started informal logic (and, or, if/then, iff, not, for all, there exists). We have seen mathematical statements, truth values, and how to use the connectives. We have seen truth tables, contradictions and tatutologies. We have not (formally) seen how to negate a statement (although some students in LEC0101 will have seen it, and it is not on today's quiz).

Suggested Activities

Activity 0. Check-in with the students: How is the term going so far? How do they feel about the upcoming quiz? How did Problem Set 1 go? (Remind students that they can ask/answer questions on Piazza.)

The first exercise is to warm them up and remind them how the logical connectives work.

Activity 1. Express the following statements in English in a way that someone in grade 8 would understand. Are the following statements true or false? Justify your answer with a proof.

(1) $(\exists x \in \mathbb{Z})[x^3 > 10]$

(2) $(\forall x \in \mathbb{R})[x > 0 \vee x < 0]$.

(3) $(\forall x \in \mathbb{R})[x = -2 \implies x^2 = 4]$

(4) $(\forall x \in \mathbb{R})[x^2 = 4 \implies x = -2]$

(5) $(\forall x \in \mathbb{R})[x^2 = 4 \iff (x = -2 \vee x = 2)]$

The next one is designed to help them distinguish between the order of quantifiers. The last question is "easy" if they understand quantifiers, and impossible if they don't. Find the people who think it's impossible (because they are stuck).

Activity 2. Express the following statements in English in a way that someone in grade 8 would understand. Are the following statements true or false? Justify your answer with a proof.

(1) $(\exists x \in \mathbb{N})(\exists y \in \mathbb{N})[x = -y]$

(2) $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})[x < y \implies x^2 < y^2]$.

(3) $(\forall x \in \mathbb{N})(\exists y \in \mathbb{Z})[x + y = 0]$

(4) $(\exists y \in \mathbb{Z})(\forall x \in \mathbb{N})[x + y = 0]$

(5) $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(\forall z \in \mathbb{R})(\exists a \in \mathbb{R})[a + x = y + z]$

The next activity is to strengthen their understanding of vacuous truth. (We didn't mention this explicitly in LEC0101.)

Activity 3. The following activities will help you understand "vacuous truth".

(1) Is the following true: "Every vowel in the word Toronto is an o.". How would you prove it?

(2) Is the following true: "Every consonant in the word Toronto is a 't'.". How would you prove it?

(3) Is the following true: "Every Greek letter in the word Toronto is a Σ."? How would you prove it?

(4) Is the following true: "If x is a consonant in the word Toronto, then x is a 't'." How would you prove it?

(5) Is the following true: "If x is a Greek letter in the word Toronto, then x is a Σ."? How would you prove it?

(6) Is the following true: $(\forall x \in \mathbb{R})[x^2 < 0 \implies x > 0]$?

This question is from the textbook, and is one that some students were asking about:

Activity 4 (Question 3.7.5 from the textbook)**.** Let $P(x)$ be the assertion "x is positive", and let $Q(x)$ be the assertion "$x^2 > x$".

(1) Is the statement $(\forall x \in \mathbb{R})[P(x) \implies Q(x)]$ true or false? Why?

(2) Is the statement $(\forall x \in \mathbb{R})[P(x)] \implies (\forall x \in \mathbb{R})[Q(x)]$ true or false? Why?

Tutorial for Week 3 - Handout

By the end of this tutorial, students should be able to:

(1) Translate a mathematical statement into English.

(2) Identify when an implication is vacuously true.

(3) Prove a statement involving quantifiers.

Suggested Activities

Activity 1. Express the following statements in English in a way that someone in grade 8 would understand. Are the following statements true or false? Justify your answer with a proof.

(1) $(\exists x \in \mathbb{Z})[x^3 > 10]$

(2) $(\forall x \in \mathbb{R})[x > 0 \vee x < 0]$.

(3) $(\forall x \in \mathbb{R})[x = -2 \implies x^2 = 4]$

(4) $(\forall x \in \mathbb{R})[x^2 = 4 \implies x = -2]$

(5) $(\forall x \in \mathbb{R})[x^2 = 4 \iff (x = -2 \vee x = 2)]$

Activity 2. Express the following statements in English in a way that someone in grade 8 would understand. Are the following statements true or false? Justify your answer with a proof.

(1) $(\exists x \in \mathbb{N})(\exists y \in \mathbb{N})[x = -y]$

(2) $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})[x < y \implies x^2 < y^2]$.

(3) $(\forall x \in \mathbb{N})(\exists y \in \mathbb{Z})[x + y = 0]$

(4) $(\exists y \in \mathbb{Z})(\forall x \in \mathbb{N})[x + y = 0]$

(5) $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(\forall z \in \mathbb{R})(\exists a \in \mathbb{R})[a + x = y + z]$

Activity 3. The following activities will help you understand "vacuous truth".

(1) Is the following true: "Every vowel in the word Toronto is an o.". How would you prove it?

(2) Is the following true: "Every consonant in the word Toronto is a 't'.". How would you prove it?

(3) Is the following true: "Every Greek letter in the word Toronto is a Σ."? How would you prove it?

(4) Is the following true: "If x is a consonant in the word Toronto, then x is a 't'." How would you prove it?

(5) Is the following true: "If x is a Greek letter in the word Toronto, then x is a Σ."? How would you prove it?

(6) Is the following true: $(\forall x \in \mathbb{R})[x^2 < 0 \implies x > 0]$?

Activity 4 (Question 3.7.5 from the textbook)**.** Let $P(x)$ be the assertion "x is positive", and let $Q(x)$ be the assertion "$x^2 > x$".

(1) Is the statement $(\forall x \in \mathbb{R})[P(x) \implies Q(x)]$ true or false? Why?

(2) Is the statement $(\forall x \in \mathbb{R})[P(x)] \implies (\forall x \in \mathbb{R})[Q(x)]$ true or false? Why?

Tutorial for Week 4 - TA Version

By the end of this tutorial, students should be able to:

(1) Distinguish between the converse, the contrapositive, and the negation of an implication.

(2) Prove a result similar to the pigeonhole principle using proof by contradiction.

(3) Rewrite a proof presented as a proof by contradiction that is really a direct proof.

(4) Decide when to use a direct proof, proof by contrapositive, or proof by contradiction.

Overview Last week we had Quiz 1, and so the week only had two weeks of lecture. We covered Negations and proof techniques (direct, contrapositive, contradiction).

Problem Set 2 is due at the end of the week. Some of the questions are designed to help them with this.

Suggested Activities

Activity 0. Check-in with the students: How did Quiz 2 go? How is Problem Set 2 going? (You might take some time to brainstorm strategies for students who didn't do as well as they hoped.)

The following exercise is to help them distinguish between the converse, the contrapositive, and the negation of an implication. (Students often confuse these concepts.)

$P \implies Q$ is logically equivalent to its contrapositive and the iterated converse at the end (since there are an even number of converses). The converse is equivalent to the converse of the contrapositve, and the contrapositive of the converse. Neither are equivalent to the negation.

Activity 1. Which of the following statements are logically equivalent? Express them in logical symbols.

(1) $P \implies Q$.

(2) The converse of $P \implies Q$.

(3) The contrapositive of $P \implies Q$.

(4) The negation of $P \implies Q$.

(5) The converse of the contrapositve of $P \implies Q$.

(6) The contrapositive of the converse of $P \implies Q$.

(7) The converse of the converse of ... [2020 times] ... of $P \implies Q$.

Q4 on PS2 uses the pigeonhole principle. The following will help them understand how to prove the Pigeonhole principle. If you want to push them, ask them to generalize this problem. ("Does this problem only apply to 2020? How would this problem change if it was assigned next year? What is the general result?")

Activity 2. (Exercise from MAT102 Winter 2020) Prove by contradiction: If $a_1, a_2, \ldots, a_{2020} \in \mathbb{N}$ and $a_1 + a_2 + \ldots + a_{2020} \leq 2022$ then $(\forall 1 \leq i \leq 2020)[a_i \leq 3]$.

Activity 3. It is common for beginners to overuse proof by contradiction when it really isn't needed. This can make the argument hard to understand.

Rewrite the following proof so that it is a direct proof, and not a proof by contradiction.

Theorem 5. *Let $x \in \mathbb{N}$. If $x > 1$, then $x^3 + 1$ is composite.*

Proof. Assume that $x > 1$ and for a contradiction assume that $x^3 + 1$ is not composite. Since $x > 1$ we must have $x^3 + 1 > 1$. This means that $x^3 + 1$ is prime.

You can see that $(x + 1)(x^2 - x + 1) = x^3 + 1$, and both factors are integers. Since $x > 1$ we see that $1 < x + 1 < x^3 + 1$. Therefore $x^3 + 1$ is not prime. This contradicts that $x^3 + 1$ is prime. $\Rightarrow\Leftarrow$. □

Here's what we expect:

Proof. Assume that $x > 1$.

You can see that $(x + 1)(x^2 - x + 1) = x^3 + 1$, and both factors are integers. Since $x > 1$ we see that $1 < x + 1 < x^3 + 1$. Therefore $x^3 + 1$ is composite. □

One big tell that it isn't a "real" proof by contradiction is that the assumption "$\neg Q$" was never really used.

Activity 4. For each of the following statements decide which proof technique (direct, contrapositive, contradiction) you should try first.

(1) Let a, b, c be integers. If $a|b$ and $b|c$ then $a|c$.

(2) Let $x \in \mathbb{Z}$. If $6 \nmid x$ then $3 \nmid x$.

(3) Let $m, n \in \mathbb{Z}$. If $m^2 + n^2$ is divisible by 4, then both m and n are even numbers. (This is textbook 3.7.30.b)

(4) There are no rational solutions to $x + x^3 = 1$.

Tutorial for Week 4 - Handout

By the end of this tutorial, students should be able to:

(1) Distinguish between the converse, the contrapositive, and the negation of an implication.

(2) Prove a result similar to the pigeonhole principle using proof by contradiction.

(3) Rewrite a proof presented as a proof by contradiction that is really a direct proof.

(4) Decide when to use a direct proof, proof by contrapositive, or proof by contradiction.

Suggested Activities

Activity 1. Which of the following statements are logically equivalent? Express them in logical symbols.

(1) $P \implies Q$.

(2) The converse of $P \implies Q$.

(3) The contrapositive of $P \implies Q$.

(4) The negation of $P \implies Q$.

(5) The converse of the contrapositve of $P \implies Q$.

(6) The contrapositive of the converse of $P \implies Q$.

(7) The converse of the converse of ... [2020 times] ... of $P \implies Q$.

Activity 2. (Exercise from MAT102 Winter 2020) Prove by contradiction: If $a_1, a_2, \ldots, a_{2020} \in \mathbb{N}$ and $a_1 + a_2 + \ldots + a_{2020} \leq 2022$ then $(\forall 1 \leq i \leq 2020)[a_i \leq 3]$.

Activity 3. It is common for beginners to overuse proof by contradiction when it really isn't needed. This can make the argument hard to understand.

Rewrite the following proof so that it is a direct proof, and not a proof by contradiction.

Theorem 6. *Let $x \in \mathbb{N}$. If $x > 1$, then $x^3 + 1$ is composite.*

Proof. Assume that $x > 1$ and for a contradiction assume that $x^3 + 1$ is not composite. Since $x > 1$ we must have $x^3 + 1 > 1$. This means that $x^3 + 1$ is prime.

You can see that $(x + 1)(x^2 - x + 1) = x^3 + 1$, and both factors are integers. Since $x > 1$ we see that $1 < x + 1 < x^3 + 1$. Therefore $x^3 + 1$ is not prime. This contradicts that $x^3 + 1$ is prime. $\Rightarrow\Leftarrow$. □

Activity 4. For each of the following statements decide which proof technique (direct, contrapositive, contradiction) you should try first.

(1) Let a, b, c be integers. If $a|b$ and $b|c$ then $a|c$.

(2) Let $x \in \mathbb{Z}$. If $3 \nmid x$ then $6 \nmid x$.

(3) Let $m, n \in \mathbb{Z}$. If $m^2 + n^2$ is divisible by 4, then both m and n are even numbers. (This is textbook 3.7.30.b)

(4) There are no rational solutions to $x + x^3 = 1$.

Tutorial for Week 5 - TA Version

By the end of this tutorial, students should be able to:

(1) Find a counterexample to a false set "identity".

(2) Prove a set identity by definition unwinding.

(3) Compute various things related to functions (domain, range, codomain, image).

Overview The students just had PS2 due last Friday. Their Quiz 2 is happening on Tuesday; many of them will be focused on that. This tutorial will have some overlap with the Quiz.

Suggested Activities

Activity 0. Check-in with the students: How is the term going so far? How do they feel about the upcoming quiz? How did Problem Set 2 go?

Activity 1. Let A, B, C be sets. Provide a counterexample for each of the following false set identities. (Draw a Venn diagram first.)

(1) $A \setminus (B \cup C) = (A \setminus B) \cup (A \setminus C)$

(2) $(A \cap B) \setminus C = A \cap C^c$.

Activity 2. Prove the following set identities where A, B, C are sets.

(1) $(A \cap B) \subset (A \cup B)$

(2) $A \cap B \cap C = (A \cap B) \cap (B \cap C) \cap (A \cap C)$

(3) $A \subseteq B$ if and only if $A \setminus B = \emptyset$.

Activity 3. List out, or draw a picture of the following sets.

(1) $\{1, 2\} \times \{2, 3\}$

(2) $\mathbb{R} \times [0, 1]$

(3) $[0, 1] \times \mathbb{R}$

(4) $([1, 2] \cup [3, 4]) \times ([-1, 0] \cup [2, 3])$

Activity 4. For each of the following functions, find its domain, codomain, and range.

(1) $f : \{1, 2, 3\} \to [-3, -1]$ given by $f(x) = -x$.

(2) The function that takes in a person in the tutorial and outputs their birth year.

(3) A function that takes in a block of code and determines if it compiles or not.

Tutorial for Week 5 - Handout

By the end of this tutorial, students should be able to:

(1) Find a counterexample to a false set "identity".

(2) Prove a set identity by definition unwinding.

(3) Compute various things related to functions (domain, range, codomain, image).

Suggested Activities

Activity 1. Let A, B, C be sets. Provide a counterexample for each of the following false set identities. (Draw a Venn diagram first.)

(1) $A \setminus (B \cup C) = (A \setminus B) \cup (A \setminus C)$

(2) $(A \cap B) \setminus C = A \cap C^c$.

Activity 2. Prove the following set identities where A, B, C are sets.

(1) $(A \cap B) \subset (A \cup B)$

(2) $A \cap B \cap C = (A \cap B) \cap (B \cap C) \cap (A \cap C)$

(3) $A \subseteq B$ if and only if $A \setminus B = \emptyset$.

Activity 3. List out, or draw a picture of the following sets.

(1) $\{1, 2\} \times \{2, 3\}$

(2) $\mathbb{R} \times [0, 1]$

(3) $[0, 1] \times \mathbb{R}$

(4) $([1, 2] \cup [3, 4]) \times ([-1, 0] \cup [2, 3])$

Activity 4. For each of the following functions, find its domain, codomain, and range.

(1) $f : \{1, 2, 3\} \to [-3, -1]$ given by $f(x) = -x$.

(2) The function that takes in a person in the tutorial and outputs their birth year.

(3) A function that takes in a block of code and determines if it compiles or not.

Tutorial for Week 6 - TA Version

By the end of this tutorial, students should be able to:

(1) Identify structure in the the power set of $\{1, 2, 3\}$.

(2) Prove that a relation is reflexive, symmetric, or transitive, or provide counterexamples.

(3) Solve an abstract problem about equivalence relations by examining small examples.

Overview PS3 is due at the end of the week. It is difficult for two reasons: (1) we introduce a lot of new concepts on PS3, and (2) the final question is very abstract. This Tutorial will help with both of those things.

Suggested Activities

Activity 0. Check-in with the students: How is Problem Set 3 going? Where are they stuck? (Remind students that they can ask/answer questions on Piazza or in office hours)

This first question is related to Q1 on PS3. Do as many questions as it takes for it to click with them. You don't need to do everything.

Activity 1. This question is related to Q1 on PS3 and will help you understand the structure of the power set.

Recall that $\binom{n}{r}$ is the number of ways to choose r elements from an n element set.

(1) Write down all subsets of $\{1, 2, 3\}$, and sort them into sets with 3 elements, 2 elements, 1 element, and 0 elements.

(2) How many subsets of $\{1, 2, 3\}$ are there with exactly 2 elements. Explain why this is $\binom{3}{2}$.

(3) How many subsets of $\{1, 2, 3\}$ are there with exactly 1 element. Explain why this is $\binom{3}{1}$.

(4) Justify why the set $\{1, 2, \ldots, 2020\}$ has exactly $\binom{2020}{r}$ many subsets with r elements.

(5) Use part 4 to explain why $\binom{2020}{0} = 1$ and $\binom{2020}{2020} = 1$.

(6) Use parts 2, 3, and 4, to explain why $\binom{2020}{r} = \binom{2020}{2020-r}$.

(7) Use part 1 to prove that $\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 8$.

This exercise is to play around with how relations work. The two different notations (R as a list of pairs, and the $\sim$ notation) are intentional.

Activity 2. For each of the following relations, either prove that it has each the given properties or provide a counterexample.

(1) $X = \{1, 2, 3, 4\}$, $R = \{(1, 1), (2, 3), (3, 2), (3, 3), (3, 4), (4, 3), (4, 4)\}$, transitive, symmetric and reflexive.

(2) $X = \mathbb{N}$, $a \sim b$ if and only if $a|b$, transitive, symmetric and reflexive

This activity is to help with the abstract Q4 on PS3. Emphasize the value of using examples to help cut through the fog of abstraction and new definitions. Discuss how you usually parse new (abstract) definitions, and the role of creating examples for you.

Activity 3. For PS3 Q4, the question is quite abstract. This activity is designed to help you with that abstraction.

(1) Write down two (explicit) relations R and S on the set $X = \{1, 2, 3\}$ such that $R \subseteq S$.

(2) Suppose that R and S are relations on a set X such that $R \subseteq S$.

   (a) Show that if R is reflexive, then S must be reflexive.

   (b) Show that there is an example where S is reflexive, but R isn't.

   (c) Show that there is an example where R is symmetric, but S isn't.

   (d) Show that there is an example where S is symmetric, but R isn't.

   (e) Show that there is an example where R is an equivalence relation, but S isn't.

Tutorial for Week 6 - Handout

By the end of this tutorial, students should be able to:

(1) Identify structure in the the power set of $\{1, 2, 3\}$.

(2) Prove that a relation is reflexive, symmetric, or transitive, or provide counterexamples.

(3) Solve an abstract problem about equivalence relations by examining small examples.

Suggested Activities

Activity 1. This question is related to Q1 on PS3 and will help you understand the structure of the power set.

Recall that $\binom{n}{r}$ is the number of ways to choose r elements from an n element set.

(1) Write down all subsets of $\{1, 2, 3\}$, and sort them into sets with 3 elements, 2 elements, 1 element, and 0 elements.

(2) How many subsets of $\{1, 2, 3\}$ are there with exactly 2 elements. Explain why this is $\binom{3}{2}$.

(3) How many subsets of $\{1, 2, 3\}$ are there with exactly 1 element. Explain why this is $\binom{3}{1}$.

(4) Justify why the set $\{1, 2, \ldots, 2020\}$ has exactly $\binom{2020}{r}$ many subsets with r elements.

(5) Use part 4 to explain why $\binom{2020}{0} = 1$ and $\binom{2020}{2020} = 1$.

(6) Use parts $2, 3$, and 4, to explain why $\binom{2020}{r} = \binom{2020}{2020-r}$.

(7) Use part 1 to prove that $\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 8$.

Activity 2. For each of the following relations, either prove that it has each the given properties or provide a counterexample.

(1) $X = \{1, 2, 3, 4\}$, $R = \{(1, 1), (2, 3), (3, 2), (3, 3), (3, 4), (4, 3), (4, 4)\}$, transitive, symmetric and reflexive.

(2) $X = \mathbb{N}$, $a \sim b$ if and only if $a|b$, transitive, symmetric and reflexive

Activity 3. For PS3 Q4, the question is quite abstract. This activity is designed to help you with that abstraction.

(1) Write down two (explicit) relations R and S on the set $X = \{1, 2, 3\}$ such that $R \subseteq S$.

(2) Suppose that R and S are relations on a set X such that $R \subseteq S$.

   (a) Show that if R is reflexive, then S must be reflexive.

   (b) Show that there is an example where S is reflexive, but R isn't.

   (c) Show that there is an example where R is symmetric, but S isn't.

   (d) Show that there is an example where S is symmetric, but R isn't.

   (e) Show that there is an example where R is an equivalence relation, but S isn't.

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Tutorial for Week 7 - TA Version

By the end of this tutorial, students should be able to:

(1) Use the triangle inequality to prove a bounding argument.

(2) Apply the AMGM inequality to prove new inequalities.

Overview Today's tutorial activities are meant to give practice with inequalities. (The problem set due at the end of this week is all about inequalities.) The major goal here is for them to get to the point where they can prove the n-term AMGM:

$$(\forall n \in \mathbb{N})(\forall x_1, \ldots, x_n \geq 0)\left[\frac{x_1 + \ldots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}\right]$$

and the weighted version of the AMGM:

$$(\forall x, y \geq 0)(\forall a, b \geq 0)[a + b = 1 \Rightarrow ax + by \geq x^a y^b]$$

Suggested Activities

Activity 0. Check-in with the students: How was their break? How is Problem Set 4 going? What are they going to do the same/differently in this half of the term?

Activity 1. Bounding arguments. Use the triangle inequality to find a number $M \in \mathbb{R}$ such that:

$$(\forall x \in [-2, 0])[|-x^3 - 3x^2 + 6x + 1| \leq M]$$

It is known (e.g. through calculus) that the smallest M that works is 15. Explain why your M is larger than 15.

Proof.

$$\begin{aligned} |-x^3 - 3x^2 + 6x + 1| &\leq |-x^3| + |3x^2| + |-6x| + |1| \\ &\leq |x^3| + 3|x^2| + 6|x| + 1 \\ &\leq |x|^3 + 3|x|^2 + 6|x| + 1 \\ &\leq 2^3 + 3 \cdot 2^2 + 6 \cdot 2 + 1 = 33 \end{aligned}$$

Note that the largest possible value of $|x|, |x|^2$ and $|x|^3$ is when $x = -2$. The reason this value is less than 15 is because we are ignoring all possible cancellation of the terms by using the triangle inequality. The number 15 takes the cancellation into account. □

Activity 2. Use AMGM to prove:

$$(\forall x \geq 0)[x(x + 2) \leq (x + 1)^2]$$

Proof. Note that $x + 1$ is the average (arithmetic mean) of x and $x + 2$. Also, both terms are non-negative. Therefore:

$$\sqrt{x(x + 2)} \leq \frac{x + (x + 2)}{2} = x + 1$$

Squaring both sides gives (since both sides are non-negative):

$$x(x + 2) \leq (x + 1)^2$$

□

You can prompt them to think about whether this inequality is true for <u>all</u> $x \in \mathbb{R}$. It is! However, special attention is required when $-2 < x < 0$ because the AMGM no longer applies (but obviously the LHS will be negative while the RHS is positive).

Activity 3. Use AMGM (3 times!) to prove: $\forall x, y, z, w \geq 0$

$$\frac{x+y+z+w}{4} \geq \sqrt[4]{xyzw}$$

Discuss what other versions of AMGM can be proved using this technique. (Notably this works for $n =$ a power of 2.)

Proof. Applying AMGM to x, y gives:

$$\frac{x+y}{2} \geq \sqrt{xy}.$$

Applying AMGM to z, w gives:

$$\frac{z+w}{2} \geq \sqrt{zw}.$$

Applying AMGM to $\frac{x+y}{2}$ and $\frac{z+w}{2}$ gives:

$$\frac{\frac{x+y}{2} + \frac{z+w}{2}}{2} \geq \sqrt{\frac{x+y}{2}\frac{z+y}{2}} \geq \sqrt{\sqrt{xy}\sqrt{xy}} = \sqrt[4]{xyzw}.$$

□

Activity 4. Use the previous result to prove: $\forall x, y \geq 0$

$$\frac{x}{4} + \frac{3y}{4} \geq \sqrt[4]{xy^3}$$

Discuss that this is a special version of the weighted AMGM. The LHS is an average, but it's not an equal average. We add weight to the y term.

Activity 5. Use AMGM for 4 terms, with $w = \sqrt[3]{xyz}$ to prove: $\forall x, y, z \geq 0$

$$\frac{x+y+z}{3} \geq \sqrt[3]{xyz}$$

This one is challenging and will require you to give hints.

Proof. Applying the 4 term AMGM gives:

$$\frac{x+y+z+\sqrt[3]{xyz}}{4} \geq \sqrt[4]{xyz\sqrt[3]{xyz}}$$

The RHS becomes:

$$\sqrt[4]{xyz\sqrt[3]{xyz}} = \sqrt[4]{(xyz)^{\frac{4}{3}}} = \sqrt[3]{xyz}$$

So then we have:

$$\frac{x+y+z+\sqrt[3]{xyz}}{4} \geq \sqrt[3]{xyz}$$

Multiplying both sides by $\frac{4}{3}$ gives us:

$$\frac{x+y+z+\sqrt[3]{xyz}}{3} \geq \frac{4}{3}\sqrt[3]{xyz}$$

And moving the $\frac{\sqrt[3]{xyz}}{3}$ to the RHS gives us our desired inequality. □

The following three activities are challenges for interested students. You don't need to discuss these with the whole tutorial.

Activity 6. Challenge. Generalize your previous argument to show that if you know AMGM for n terms is true (and $n > 2$), then you can prove AMGM for $n-1$ terms.

Activity 7. Challenge. Combine everything you've done here to conclude that for any $n \in \mathbb{N}$ the AMGM for n terms is true.

Idea: Prove it for all powers of 2. Then use the fact that you can step down n from the previous activity.

Activity 8. Challenge. Let $n \in \mathbb{N}$. Suppose that $a_1, \ldots, a_n$ are positive rationals such that $a_1 + \ldots + a_n = 1$. Prove, using the previous result, that $\forall x_1, \ldots, x_n \geq 0$ that

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n \geq x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}.$$

This is the weighted AMGM (with rational weights).

Idea: Find a common denominator for the weights, and then this becomes a special case of the previous activity. This is a generalization of activity 3.

Tutorial for Week 7 - Handout

By the end of this tutorial, students should be able to:

(1) Use the triangle inequality to prove a bounding argument.

(2) Apply the AMGM inequality to prove new inequalities.

Suggested Activities

Activity 1. Bounding arguments. Use the triangle inequality to find a number $M \in \mathbb{R}$ such that:

$$(\forall x \in [-2, 0])[|-x^3 - 3x^2 + 6x + 1| \leq M]$$

It is known (e.g. through calculus) that the smallest M that works is 15. Explain why your M is larger than 15.

Activity 2. Use AMGM to prove:

$$(\forall x \geq 0)[x(x+2) \leq (x+1)^2]$$

Activity 3. Use AMGM (3 times!) to prove: $\forall x, y, z, w \geq 0$

$$\frac{x+y+z+w}{4} \geq \sqrt[4]{xyzw}$$

Hint:

$$\frac{\frac{x+y}{2} + \frac{z+w}{2}}{2} = \frac{x+y+z+w}{4}.$$

Activity 4. Use the previous result to prove: $\forall x, y \geq 0$

$$\frac{x}{4} + \frac{3y}{4} \geq \sqrt[4]{xy^3}$$

Activity 5. Use AMGM for 4 terms, with $w = \sqrt[3]{xyz}$ to prove: $\forall x, y, z \geq 0$

$$\frac{x+y+z}{3} \geq \sqrt[3]{xyz}$$

Activity 6. Challenge. Generalize your previous argument to show that if you know AMGM for n terms is true (and $n > 2$), then you can prove AMGM for $n-1$ terms.

Activity 7. Challenge. Combine everything you've done here to conclude that for any $n \in \mathbb{N}$ the AMGM for n terms is true.

Activity 8. Challenge. Let $n \in \mathbb{N}$. Suppose that $a_1, \ldots, a_n$ are positive rationals such that $a_1 + \ldots + a_n = 1$. Prove, using the previous result, that $\forall x_1, \ldots, x_n \geq 0$ that

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n \geq x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}.$$

This is the weighted AMGM (with rational weights).

Tutorial for Week 8 - TA Version

By the end of this tutorial, students should be able to:

(1) Generalize a statement about a particular number, to a statement $P(n)$.

(2) Improve the presentation of a proof by induction.

(3) Apply the idea of induction to create a new version of induction.

Overview The students have a quiz today covering equivalence relations.

This tutorial will allow students to practice writing proofs by induction. At this stage students will have seen simple induction, but maybe not much else.

Suggested Activities

Activity 0. Check in with the students. How did PS4 go? Any questions about Quiz 3 today?

Activity 1. For each of the following facts, generalize them to statements $P(n)$. Does your n represent something in these statements (i.e. number of terms, number of sides, etc.)? For what $n \in \mathbb{N}$ is your statement true?

(1) There are $4 \cdot 3 \cdot 2 \cdot 1$ different ways to order 4 people.

- $P(n) =$ "There are $n!$ ways to order n people"
- n represents the number of people,
- $p(n)$ is true for all $n \in \mathbb{N}$ starting with $n = 1$.

(2) $2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 = 8 \cdot 9$.

- $P(n) =$ "$2 + 4 + \ldots + 2n = n(n + 1)$"
- n represents the number of terms.
- $p(n)$ is true for all $n \in \mathbb{N}$ starting with $n = 1$.

(3) $1 + \frac{7}{\pi} \leq (1 + \frac{1}{\pi})^7$.

- $P(n) =$ "$1 + \frac{n}{\pi} \leq (1 + \frac{1}{\pi})^n$"
- n represents the number of factors.
- $p(n)$ is true for all $n \in \mathbb{N}$ starting with $n = 1$.

(4) $\forall x, y, z \in \mathbb{R}$ we have $\dfrac{x + y + z}{3} \geq \sqrt[3]{xyz}$.

- $P(n) =$ "$\forall x_1, \ldots, x_n \in \mathbb{R}, \dfrac{1}{n} \displaystyle\sum_{i=1}^{n} x_i \geq \sqrt[n]{\prod_{i=1}^{n} x_i}$".
- n represents the number of terms, or the dimension of the AMGM.
- $p(n)$ is true for all $n \in \mathbb{N}$ starting with $n = 1$.

(5) The (inner) angles of a triangle always add up to 180 degrees. The (inner) angles of a pentagon always add up to 540 degrees.

- $P(n) =$ "The inner angles of an n-gon always add up to $180(n - 2)$ degrees."
- n represents the number of sides of the polygon.

- $p(n)$ is true for all $n \in \mathbb{N}$ starting with $n = 3$.

Here is a common grading scheme for the induction questions:

- 1pt: Stated the P(n) explicitly and correctly.
- 1pt: Proved the base case explicitly and correctly. (Do not award any points if the student starts with the conclusion, and then derives a true statement.)
- 1pt: The induction hypothesis was explicitly assumed for a particular $n \in \mathbb{N}$, and its use was pointed out (correctly).
- 1pt: The structure of the proof of the inductive step was correct. (Do not award any points if the student starts with the conclusion, and then derives a true statement.)

Activity 2. Use the above grading scheme to grade the following proof of the fact that "For all $n \in \mathbb{N}$, $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$."

Then, improve the proof so that it is graded 4/4.

Rough proof. Let $P(n)$ be the statement "For all $n \in \mathbb{N}$, $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$".

For $n = 1$, Note that

$$\begin{aligned} 1 + 2^1 &= 2^2 - 1 \\ \Rightarrow 1 + 2 &= 3 - 1 \\ \Rightarrow 3 &= 3 \checkmark \end{aligned}$$

Now assume $P(n)$ for all $n \in \mathbb{N}$. So

$$\begin{aligned} 1 + 2 + 4 + \ldots + 2^n + 2^{n+1} &= 2^{n+1} - 1 + 2^{n+1} \\ &= 2 \cdot 2^{n+1} - 1 \\ &= 2^{n+2} - 1. \end{aligned}$$

Therefore $P(n + 1)$ is true, as desired. □

The grading scheme says to award that proof 1/4 (the one point is for the correct structure of the inductive step). Here is the improved (4/4) proof:

Improved proof. Let $P(n)$ be the statement "$1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$".

For $n = 1$, Note that

$$1 + 2^1 = 3 = 4 - 1 = 2^{1+1} - 1.$$

Now assume $P(n)$ for some $n \in \mathbb{N}$. So

$$\begin{aligned} 1 + 2 + 4 + \ldots + 2^n + 2^{n+1} &= 2^{n+1} - 1 + 2^{n+1} && \text{by the IH} \\ &= 2 \cdot 2^{n+1} - 1 \\ &= 2^{n+2} - 1. \end{aligned}$$

Therefore $P(n + 1)$ is true, as desired. □

Activity 3. Martine is playing a board game that uses meeple as a resource. She knows that:

A1. If she can win the game with n meeple, then she can also win with $n - 3$ meeple (as long as $n - 3 \geq 0$).

A2. If she can win the game with n meeple, then she can also win with $n-5$ meeple (as long as $n-5 \geq 0$).

A3. She knows how to win the game with 20 meeple.

What are all the amounts of meeple that Martine can win with? How does this change if (A3) is changed to (A3*): She knows how to win the game with 2020 meeple (instead of 20)?

Warning. Some students will be deeply confused that Martine can win with 14 meeple. (Their confusion comes from thinking that (A1) and (A2) only apply to $n = 20$.) It's worth discussing what (A1) and (A2) actually say, and where the hidden "$\forall n$" should go.

Proof. We observe that:

Starting meeple	**Result**	**Proof**
20	Win	20
19		
18		
17	Win	20-3
16		
15	Win	20-5
14	Win	20-3-3
13		
12	Win	20-5-3
11	Win	20-3-3-3
10	Win	20-5-5

Everything from 0 to 9 will be three less than a winning position, so by $A1$ she can win with all those as well.

If we use (A3*) instead of (A3), then she can win from all positions except $2020-1$, $2020-2$, $2020-4$, and $2020-7$.

□

Tutorial for Week 8 - Handout

By the end of this tutorial, students should be able to:

(1) Generalize a statement about a particular number, to a statement $P(n)$.

(2) Improve the presentation of a proof by induction.

(3) Apply the idea of induction to create a new version of induction.

Suggested Activities

Activity 1. For each of the following facts, generalize them to statements $P(n)$. Does your n represent something in these statements (i.e. number of terms, number of sides, etc.)? For what $n \in \mathbb{N}$ is your statement true?

(1) There are $4 \cdot 3 \cdot 2 \cdot 1$ different ways to order 4 people.

(2) $2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 = 8 \cdot 9$.

(3) $1 + \frac{7}{\pi} \leq (1 + \frac{1}{\pi})^7$.

(4) $\forall x, y, z \in \mathbb{R}$ we have $\dfrac{x+y+z}{3} \geq \sqrt[3]{xyz}$.

(5) The (inner) angles of a triangle always add up to 180 degrees. The (inner) angles of a pentagon always add up to 540 degrees.

Here is a common grading scheme for the induction questions:

- 1pt: Stated the P(n) explicitly and correctly.
- 1pt: Proved the base case explicitly and correctly. (Do not award any points if the student starts with the conclusion, and then derives a true statement.)
- 1pt: The induction hypothesis was explicitly assumed for a particular $n \in \mathbb{N}$, and its use was pointed out (correctly).
- 1pt: The structure of the proof of the inductive step was correct. (Do not award any points if the student starts with the conclusion, and then derives a true statement.)

Activity 2. Use the above grading scheme to grade the following proof of the fact that "For all $n \in \mathbb{N}$, $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$."

Then, improve the proof so that it is graded 4/4.

Rough proof. Let $P(n)$ be the statement "For all $n \in \mathbb{N}$, $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$".

For $n = 1$, Note that

$$\begin{aligned} &1 + 2^1 = 2^2 - 1 \\ &\Rightarrow 1 + 2 = 3 - 1 \\ &\Rightarrow 3 = 3 \checkmark \end{aligned}$$

Now assume $P(n)$ for all $n \in \mathbb{N}$. So

$$\begin{aligned} 1 + 2 + 4 + \ldots + 2^n + 2^{n+1} &= 2^{n+1} - 1 + 2^{n+1} \\ &= 2 \cdot 2^{n+1} - 1 \\ &= 2^{n+2} - 1. \end{aligned}$$

Therefore $P(n+1)$ is true, as desired. $\square$

Activity 3. Martine is playing a board game that uses meeple as a resource. She knows that:

A1. If she can win the game with n meeple, then she can also win with $n-3$ meeple (as long as $n-3 \geq 0$).

A2. If she can win the game with n meeple, then she can also win with $n-5$ meeple (as long as $n-5 \geq 0$).

A3. She knows how to win the game with 20 meeple.

What are all the amounts of meeple that Martine can win with? How does this change if (A3) is changed to (A3*): She knows how to win the game with 2020 meeple (instead of 20)?

Tutorial for Week 9 - TA Version

By the end of this tutorial, students should be able to:

(1) Decide which version of induction to apply in a given situation.

(2) Prove that a given explicit formula generates a given recursively defined sequence.

(3) Translate a proof by induction into a construction.

Overview We are now finished induction, and moving on to injections, surjections, bijections. They have PS5 due on Friday; it's all about induction. This tutorial should help them with it.

Many of them will understand the mechanics of writing a proof by induction, but they don't understand what it is actually doing. The third activity has them extract a method from a proof by induction and apply it to a specific number. This will force them to understand what the proof is actually doing.

Suggested Activities

Activity 0. Check-in with the students: How is the second term going so far? How is Problem Set 5 going?

Activity 1. What are the 5 different variations on induction that we looked at? State them. For each of the following facts, choose the variation that is most appropriate. (**Warning**. Not all 5 variations will show up in this activity!)

(1) Whenever n is a natural number, then $\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$ is an integer.

(2) $10^n - 1$ is divisible by 11 for every even natural number n.

(3) For large enough natural numbers n, we must have $n^3 < 3^n$.

(4) Let a_n be a sequence defined recursively as $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 3$. Given that a_1, a_2 are odd, show that all a_n are odd.

(5) Every natural number n can be written in base 10 in only one way.

The five variations are:

(1) Simple (or usual or weak) induction.

(2) Induction starting at a different base case.

(3) Induction on the evens.

(4) Induction on the odds.

(5) Strong induction.

For Q4, it is acceptable to say that this is strong induction, although it's closer to simple induction. You can delay discussing this until after the final activity.

Activity 2. Let $a_1 = 1$, and let $a_{n+1} = 3a_n + 1$ for $n \in \mathbb{N}$. Compute a_2, a_3, a_4 using this recursive definition.

Prove that, for all $n \in \mathbb{N}$ that

$$a_n = \frac{3^n - 1}{2}$$

Proof. Note that $\frac{3^1-1}{2} = \frac{2}{2} = 1 = a_1$.

Now, suppose that $a_n = \frac{3^n-1}{2}$ for a particular $n \in \mathbb{N}$.

Note that

$$\begin{aligned} a_{n+1} &= 3a_n + 1 && \text{By definition} \\ &= 3\frac{3^n - 1}{2} + 1 && \text{By IH} \\ &= \frac{3 \cdot 3^n - 3 + 2}{2} \\ &= \frac{3^{n+1} - 1}{2} \end{aligned}$$

As desired. □

Note. Many students mistakenly believe that "$\frac{3^n-1}{2}$" is the IH. Challenge them on this by asking them about mathematical statements. (Is it true or false? Is $7n$ true?)

Activity 3. Generalize your previous proof to find an explicit formula for the sequence defined by: $b_1 = 1$ and $b_{n+1} = 13b_n + 1$.

Line 3 of the previous proof tells us how to generalize this. The formula will be:

$$b_n = \frac{13^n - 1}{12}.$$

In general, if $c \in \mathbb{R}$ and $c > 1$, then the recursive formula $a_1 = 1$ and $a_{n+1} = ca_n + 1$ will have description:

$$a_n = \frac{c^n - 1}{c - 1}$$

You can push stronger students to look for that formula. That fraction might look familiar as it is the geometric sum formula.

$$\frac{c^n - 1}{c - 1} = 1 + c + c^2 + \ldots + c^{n-1}.$$

This is one possible motivation for how we came up with this formula.

Activity 4. The following is a proof that every integer can be written as a power of 2 multiplied by an odd number. For the sake of this question, call this a "parity decomposition."

(1) Before reading the proof, show that 700 has a parity decomposition.

(2) Reflect. Was your method for solving part 1 applicable to other numbers? If so, explain how you find the parity decomposition of any number.

(3) From the proof, extract a method for finding the parity decomposition of 700 and then any number.

(4) Does finding the parity decomposition of 700 involve the parity decomposition of 699?

(5) How much does the proof require that you remember to find the parity decomposition of 700?

Proof. Let $P(n)$ be the statement "n can be written as a product of a power of 2 and an odd number." We proceed by strong induction.

Base. If $n = 1$, then $1 = 2^0(1)$, and 1 is odd.

Induction. Suppose $P(1), P(2), \ldots, P(n)$ are true for a particular n.

Case 1. If $n + 1$ is odd, then $n + 1 = 2^0(n + 1)$ is the parity decomposition.

Case 2. If $n + 1$ is even, then there is a (positive) integer m with $2m = n + 1$. Since $P(m)$ is true, there are integers a and odd b such that $m = 2^a b$. Therefore

$$n + 1 = 2m = 2 \cdot 2^a b = 2^{a+1} b,$$

as desired. □

When discussing this, emphasize that this is strong induction because it looks back more than one or two steps, it looks back $n/2$ steps. If you really want to get into it, and someone is interested, we usually think of simple induction as looking back $O(1)$ steps, but strong induction is strictly more than $O(1)$.

Tutorial for Week 9 - Handout

By the end of this tutorial, students should be able to:

(1) Decide which version of induction to apply in a given situation.

(2) Prove that a given explicit formula generates a given recursively defined sequence.

(3) Translate a proof by induction into a construction.

Suggested Activities

Activity 1. What are the 5 different variations on induction that we looked at? State them. For each of the following facts, choose the variation that is most appropriate. (**Warning**. Not all 5 variations will show up in this activity!)

(1) Whenever n is a natural number, then $\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$ is an integer.

(2) $10^n - 1$ is divisible by 11 for every even natural number n.

(3) For large enough natural numbers n, we must have $n^3 < 3^n$.

(4) Let a_n be a sequence defined recursively as $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 3$. Given that a_1, a_2 are odd, show that all a_n are odd.

(5) Every natural number n can be written in base 10 in only one way.

Activity 2. Let $a_1 = 1$, and let $a_{n+1} = 3a_n + 1$ for $n \in \mathbb{N}$. Compute a_2, a_3, a_4 using this recursive definition.

Prove that, for all $n \in \mathbb{N}$ that

$$a_n = \frac{3^n - 1}{2}$$

Activity 3. Generalize your previous proof to find an explicit formula for the sequence defined by: $b_1 = 1$ and $b_{n+1} = 13b_n + 1$.

Activity 4. The following is a proof that every integer can be written as a power of 2 multiplied by an odd number. For the sake of this question, call this a "parity decomposition."

(1) Before reading the proof, show that 700 has a parity decomposition.

(2) Reflect. Was your method for solving part 1 applicable to other numbers? If so, explain how you find the parity decomposition of any number.

(3) From the proof, extract a method for finding the parity decomposition of 700 and then any number.

(4) Does finding the parity decomposition of 700 involve the parity decomposition of 699?

(5) How much does the proof require that you remember to find the parity decomposition of 700?

Proof. Let $P(n)$ be the statement "n can be written as a product of a power of 2 and an odd number." We proceed by strong induction.

Base. If $n = 1$, then $1 = 2^0(1)$, and 1 is odd.

Induction. Suppose $P(1), P(2), \ldots, P(n)$ are true for a particular n.

Case 1. If $n + 1$ is odd, then $n + 1 = 2^0(n + 1)$ is the parity decomposition.

Case 2. If $n+1$ is even, then there is a (positive) integer m with $2m = n+1$. Since $P(m)$ is true, there are integers a and odd b such that $m = 2^a b$. Therefore

$$n + 1 = 2m = 2 \cdot 2^a b = 2^{a+1} b,$$

as desired. □

Tutorial for Week 10 - TA Version

By the end of this tutorial, students should be able to:

(1) Identify a function that is not injective, and find an example why.

(2) Identify a function that is not surjective, and find an example why.

(3) Prove general facts about compositions of functions, from definitions.

Overview We've finished induction, and spent last week on functions: injections, surjections, bijections, inverses, and compositions. This material needs to be mastered, otherwise students will be unable to process what's happening in the countability section (Week 10 in class).

Activities 4-7 are adaptations of a PS3 question (the one about compressions and decreasing functions).

Suggested Activities

Activity 0. Check-in with the students: How do they feel about the quiz today? How did Problem Set 5 go?

Activity 1. Let $f : A \to B$. Which of the following two statements means that f is injective? What does the other one mean? Write the contrapositives of both implications.

(1) $\forall x_1, x_2 \in A$, if $x_1 = x_2$, then $f(x_1) = f(x_2)$.

(2) $\forall x_1, x_2 \in A$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

1. is the definition of being a function. It is vertical line test. 2. is the definition of injective. (Its contrapositive is "2-to-2", which is a better name for 1-to-1.)

Activity 2. Give specific points a, b that show that these functions are not injective:

(1) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by $f(x) = \frac{1}{x^2}$.

- Example: $a = 1$ and $b = -1$.

(2) $g : (\frac{-\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ given by $g(x) = \cos(x)$.

- Example: $a = \frac{-\pi}{4}$ and $b = \frac{\pi}{4}$.

(3) $h : \mathbb{N} \to \mathbb{R}$ given by $h(n) = (-1)^n$.

- Example: $a = 2$ and $b = 4$.

Activity 3. Give specific points y the codomain of these functions that show that these functions are not surjective:

(1) $f : \mathbb{R} \setminus \{0\} \to [0, \infty)$ given by $f(x) = \frac{1}{x^2}$.

- Example: $y = 0$, because $\frac{1}{x^2} > 0$.

(2) $g : [\frac{-\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}$ given by $g(x) = \cos(x)$.

- Example: $y = 2$, because $\cos x \leq 1 < 2$ for all $x \in \mathbb{R}$.

(3) $h : \mathbb{N} \to [-1, 1]$ given by $h(n) = (-1)^n$.

- Example: $y = 0$, because h only has outputs -1 and 1.

Activity 4. (**Review from PS3**) Show that there are 8 different functions with domain $\{1, 2, 3\}$ and codomain $\{0, 1\}$.

Proof. The functions written as $f(1)f(2)f(3)$. So for example 110 is the function $f(1) = 1, f(2) = 1, f(3) = 0$. This notation suggests the (true) fact that the collection of all functions from $\{1, 2, \dots, n\}$ to $0, 1$ is the collection of all binary strings of length n. This makes counting easier.

$$000, 001, 010, 011, 100, 101, 110, 111$$

Let them discover this fact! □

Activity 5. Show that there are 6 different surjective functions with domain $\{1, 2, 3\}$ and codomain $\{0, 1\}$.

Proof. Everything above works EXCEPT 000 and 111. This reminds us of this important principle of counting: Instead of counting the good things, you can count the bad things. □

Activity 6. Show that there are NO injective functions with domain $\{1, 2, 3\}$ and codomain $\{0, 1\}$.

Proof. This is impossible because the domain has 3 elements, but the codomain has only 2 choices. One of the choices must be repeated (by the pigeonhole principle).

Nudge them to think about how the size of the domain and the size of the codomain makes an injective function impossible. Ask them to make a more general observation. □

Activity 7. Challenge: Generalize the 3 previous exercises to functions with domain $\{1, 2, \dots, 10\}$, and codomain $\{0, 1\}$.

Proof. There are 2^n many functions, and all but 2 of them are surjective. There are never any injective functions. □

As an additional challenge, you can get them to identify a function $f : A \to \{0, 1\}$ as a subset of A. The identification is: $X = \{x \in A : f(x) = 1\}$. In this sense the function is the indicator function for the set A. This prove that the "the collection of all functions from A to $\{0, 1\}$" and "the collection of all subsets of A" have the same number of elements.

Activity 8. Let $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$. For each condition, give an example of functions $f_i : A \to B$ and $g_i : B \to A$ with the following properties, or explain why it is impossible.

(1) g_1 is the inverse of f_1.

(2) $f_2 \circ g_2(x) = g_2 \circ f_2(x)$ for all x.

(3) $f_3 \circ g_3(x) = x$ for all $x \in B$.

(4) The range of $g_4 \circ f_4$ has 2 elements, but the range of $f_4 \circ g_4$ has 1.

(5) The range of $g_5 \circ f_5$ has 3 elements, but the range of $f_5 \circ g_5$ has 2.

(1) Possible (easily).

(2) Impossible because they don't have the same domain!

(3) Possible if g_3 is the inverse of f_3.

(4) Possible: $f(1) = f(2) = 4, f(3) = 6$ and $g(4) = g(6) = 1, g(5) = 2$.

(5) tImpossible since the first condition forces both functions to be bijections. (This can be seen by contradiction.)

Tutorial for Week 10 - Handout

By the end of this tutorial, students should be able to:

(1) Identify a function that is not injective, and find an example why.

(2) Identify a function that is not surjective, and find an example why.

(3) Prove general facts about compositions of functions, from definitions.

Suggested Activities

Activity 1. Let $f : A \to B$. Which of the following two statements means that f is injective? What does the other one mean? Write the contrapositives of both implications.

(1) $\forall x_1, x_2 \in A$, if $x_1 = x_2$, then $f(x_1) = f(x_2)$.

(2) $\forall x_1, x_2 \in A$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

Activity 2. Give specific points a, b that show that these functions are not injective:

(1) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by $f(x) = \frac{1}{x^2}$.

(2) $g : (\frac{-\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ given by $g(x) = \cos(x)$.

(3) $h : \mathbb{N} \to \mathbb{R}$ given by $h(n) = (-1)^n$.

Activity 3. Give specific points y the codomain of these functions that show that these functions are not surjective:

(1) $f : \mathbb{R} \setminus \{0\} \to [0, \infty)$ given by $f(x) = \frac{1}{x^2}$.

(2) $g : [\frac{-\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}$ given by $g(x) = \cos(x)$.

(3) $h : \mathbb{N} \to [-1, 1]$ given by $h(n) = (-1)^n$.

Activity 4. (**Review from PS3**) Show that there are 8 different functions with domain $\{1, 2, 3\}$ and codomain $\{0, 1\}$.

Activity 5. Show that there are 6 different surjective functions with domain $\{1, 2, 3\}$ and codomain $\{0, 1\}$.

Activity 6. Show that there are NO injective functions with domain $\{1, 2, 3\}$ and codomain $\{0, 1\}$.

Activity 7. Challenge: Generalize the 3 previous exercises to functions with domain $\{1, 2, \ldots, 10\}$, and codomain $\{0, 1\}$.

Activity 8. Let $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$. For each condition, give an example of functions $f_i : A \to B$ and $g_i : B \to A$ with the following properties, or explain why it is impossible.

(1) g_1 is the inverse of f_1.

(2) $f_2 \circ g_2(x) = g_2 \circ f_2(x)$ for all x.

(3) $f_3 \circ g_3(x) = x$ for all $x \in B$.

(4) The range of $g_4 \circ f_4$ has 2 elements, but the range of $f_4 \circ g_4$ has 1.

(5) The range of $g_5 \circ f_5$ has 3 elements, but the range of $f_5 \circ g_5$ has 2.

Tutorial for Week 11 - TA Version

By the end of this tutorial, students should be able to:

(1) Define relative cardinality

(2) Identify when a statement about cardinality can be proved by definition unwinding.

(3) Create a strategy for catching a frog on $\mathbb{Z}$.

Overview Last week we started cardinality and countability, and proved simple results, and given many examples. This material is quite challenging for some students because they have many incorrect assumptions about the basic definitions.

Suggested Activities

Activity 0. Check-in with the students: How is Problem Set 6 going? How are they feeling about the end of the course and the final exam?

Activity 1. State the formal definitions of the following two statements, where A and B are sets:

(1) $|A| \leq |B|$

(2) $|A| = |B|$

SOLUTION

(1) There is an injection $f : A \to B$.

(2) There is a bijection $f : A \to B$.

Note that this only tells us how to measure relative cardinality between two sets. We have not (and will not) formally define $|A|$ for all sets A; informally we might think of $|A|$ as the number of elements of A, but this only makes sense when A is finite.

Activity 2. Find functions that verify the following facts:

(1) $|\{1, 2, 3\}| = |\{1, 4, 9\}|$,

(2) $|\mathbb{N}| \leq |\mathbb{Z}|$,

(3) $|[-2020, 2020]| \leq |(-2, 2)|$.

SOLUTION

(1) $f : \{1, 2, 3\} \to \{1, 4, 9\}$ defined by $f(x) = x^2$.

(2) $g : \mathbb{N} \to \mathbb{Z}$ defined by $g(x) = x$.

(3) $h : [-2020, 2020] \to (-2, 2)$ defined by $h(x) = \dfrac{x}{2020}$.

Activity 3. For each of the following statements identify if the proof will be simple definition unwinding or not. If it is simple, summarize the proof in one sentence. These statements are for all sets A, B, C.

(1) $|A| = |A|$.

(2) $|A| = |B| \Rightarrow |B| = |A|$.

(3) If $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$.

(4) If $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

(5) If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

(6) $|A| \leq |B|$ or $|B| \leq |A|$.

SOLUTION

If you want, you can discuss one or two of the definition unwinding proofs. Note that not everyone has seen CSB. The point of including here is to remind them that it is not a simple "definition unwinding" proof like the others.

(1) "The identity function is a bijection."

(2) "The inverse of a bijection is a bijection."

(3) "The composition of injections is an injection."

(4) "The composition of bijections is an bijection."

(5) This is the Cantor-Schroeder-Bernstein theorem. It is not obvious.

(6) This is not an obvious theorem and relies on the axiom of choice (which we do not discuss in this class).

Activity 4. This is related to Q4 on PS6.

This is a turn-based game played by Jr. Proofini and a Frog. It is played on $\mathbb{Z}$. Ahead of the game the Frog chooses a secret starting integer a. This integer cannot change during the game.

Each night, the frog moves one position up (from n to $n + 1$), and hides there until the next night.

Each day Jr. Proofini searches for the frog at any one integer she wants.

The game starts with the frog at position a, and Jr. Proofini can search on the first day.

(1) Give an example of how the strategy "Every day search at the integer 0." is not guaranteed to eventually find the frog.

(2) Give an example of how the strategy "Every day search at the integer 2020." is not guaranteed to eventually find the frog.

(3) Give an example of how the strategy "On day n search at the integer n." is not guaranteed to eventually find the frog.

(4) Show that if on day 3 Jr. Proofini searches space 7 and doesn't find the frog, then she can conclude that $a \neq 5$.

(5) Give a strategy that allows Jr. Proofini to conclude that $a \neq -1, 0, 1$ if she doesn't find the frog in the first three days.

SOLUTION

(1) If $a = 1$ then the frog will never end at 0.

(2) If $a = 2021$ then the frog will never end at 2020.

(3) If $a = 0$ (or really, anything other than 1).

(4) If $a = 5$, night 1 the frog is at 6, night 2 the frog is at 7, and then Jr. Proofini would have found it on day 3.

(5) Day 1, day 2 and day 3 search at position 1.

Tutorial for Week 11 - Handout

By the end of this tutorial, students should be able to:

(1) Define relative cardinality

(2) Identify when a statement about cardinality can be proved by definition unwinding.

(3) Create a strategy for catching a frog on $\mathbb{Z}$.

Activity 1. State the formal definitions of the following two statements, where A and B are sets:

(1) $|A| \leq |B|$

(2) $|A| = |B|$

Activity 2. Find functions that verify the following facts:

(1) $|\{1, 2, 3\}| = |\{1, 4, 9\}|$,

(2) $|\mathbb{N}| \leq |\mathbb{Z}|$,

(3) $|[-2020, 2020]| \leq |(-2, 2)|$.

Activity 3. For each of the following statements identify if the proof will be simple definition unwinding or not. If it is simple, summarize the proof in one sentence. These statements are for all sets A, B, C.

(1) $|A| = |A|$.

(2) $|A| = |B| \Rightarrow |B| = |A|$.

(3) If $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$.

(4) If $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

(5) If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

(6) $|A| \leq |B|$ or $|B| \leq |A|$.

Activity 4. This is related to Q4 on PS6.

This is a turn-based game played by Jr. Proofini and a Frog. It is played on $\mathbb{Z}$. Ahead of the game the Frog chooses a secret starting integer a. This integer cannot change during the game.

Each night, the frog moves one position up (from n to $n + 1$), and hides there until the next night.

Each day Jr. Proofini searches for the frog at any one integer she wants.

The game starts with the frog at position a, and Jr. Proofini can search on the first day.

(1) Give an example of how the strategy "Every day search at the integer 0." is not guaranteed to eventually find the frog.

(2) Give an example of how the strategy "Every day search at the integer 2020." is not guaranteed to eventually find the frog.

(3) Give an example of how the strategy "On day n search at the integer n." is not guaranteed to eventually find the frog.

(4) Show that if on day 3 Jr. Proofini searches space 7 and doesn't find the frog, then she can conclude that $a \neq 5$.

(5) Give a strategy that allows Jr. Proofini to conclude that $a \neq -1, 0, 1$ if she doesn't find the frog in the first three days.

Tutorial for Week 12 - TA Version

By the end of this tutorial, students should be able to:

(1) Evaluate functions whose domain is a power set.

(2) Identify countable and uncountable sets.

(3) Prove that two sets have the same cardinality.

Overview This is the final tutorial! You made it. Problem Set 6 was due last Friday, Quiz 5 is today, and the final exam is Friday August 21.

We'll wrap up our discussion on cardinality by focusing on countable and uncountable sets, as well as power sets. A warning is that students really don't understand power sets. Q1 is easy to solve if you know what the words mean, so give them an opportunity to work on it first.

For countability there are two skills: (1) being able to identify at a glance that something is countable, and (2) constructing a bijection to the naturals. Both of these skills will be built up in this tutorial.

The final activity is built up from common misunderstandings that students have about uncountable sets. Don't just give them the answers; allow them to discuss them and find the counterexamples.

Suggested Activities

Activity 0. Check-in with the students: This is the final tutorial; How did tutorials go this year? How do they feel about the upcoming quiz? How did Problem Set 6 go? How are they planning on preparing for the final exam?

The challenge of the following activity is for the students to read and use definitions involving the power set. The first 6 exercises are designed to help them unravel the definitions. Encourage them to try the first six exercises first, before you help them. Note that 1 and 4, 2 and 5, 3 and 6 are related.

Activity 1. Let $f : \mathcal{P}(\mathbb{N}) \to \mathbb{N} \cup \{\text{DNE}\}$, be defined by $f(A)$ is the minimum element of A, if $A \neq \emptyset$ and $f(\emptyset) = \text{DNE}$.

Let $g : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{N})$ be the (restriction) function $g(A) = A \cap \mathbb{N}$.

(1) Compute $f(\{2, 3, 5\})$.

(2) Compute $f(\mathbb{N})$.

(3) Compute $f(\emptyset)$.

(4) Compute $f \circ g([1.5, 4) \cup [4.5, 5.5])$.

(5) Compute $f \circ g(\mathbb{R})$.

(6) Compute $f \circ g([-2, 0])$.

(7) In words, explain what the function $f \circ g$ does.

(8) Prove that if $A \subseteq B \subseteq \mathbb{R}$ and $A \cap \mathbb{N} \neq \emptyset$, then $f \circ g(A) \geq f \circ g(B)$.

Note that in MAT102, the definition of countable is "in bijection with $\mathbb{N}$", and uncountable is "infinite and not countable". Officially, in MAT102, finite sets do not count as countable sets. There is disagreement about this in the mathematical community.

Students have two main tools for identifying uncountable sets. They have seen:

(1) $\mathbb{R}$ is uncountable (and every non-trivial interval).

(2) If A is infinite, then $\mathcal{P}(A)$ is uncountable.

Activity 2. Construct a bijection that shows that the following sets are countable.

(1) $\{0\} \cup \mathbb{N}$.

(2) $\{-2019, -2018, \dots, 1, 0\} \cup \mathbb{N}$.

(3) $\{-n : n \in \mathbb{N}\}$.

(4) $\mathbb{Z} \cup \{x + \frac{1}{2} : x \in \mathbb{Z}\}$.

(5) $\{1, 10, 100, 1000, \dots, \}$

(6) $\mathbb{N} \times \mathbb{Q}$.

Activity 3. Identify which of the following sets are finite, countable, or uncountable. Give a short explanation for your choice (a complete proof is not necessary).

(1) $\{1, 2, 3, \dots, 2019\}$.

(2) $\mathbb{Z}$.

(3) $A \cup \mathbb{R}$, where A is any set.

(4) $\mathcal{P}(\mathcal{P}(\mathcal{P}(\{0, 1\})))$.

(5) $\mathcal{P}(\mathbb{N}) \cap \mathbb{N}$.

(6) $\mathbb{Q} \cap [0, 1]$.

Activity 4. The following is a list of common misunderstandings that students have about uncountable sets. All these statements are false. For each of them find a counterexample to the claim.

(1) "If $A \subseteq \mathbb{R}$ then A is uncountable."

(2) "If $A, B \subseteq \mathbb{R}$" then $|A| = |B|$."

(3) "If A, B are uncountable, then $|A| = |B|$."

Tutorial for Week 12 - Handout

By the end of this tutorial, students should be able to:

(1) Evaluate functions whose domain is a power set.

(2) Identify countable and uncountable sets.

(3) Prove that two sets have the same cardinality.

Suggested Activities

Activity 1. Let $f : \mathcal{P}(\mathbb{N}) \to \mathbb{N} \cup \{\text{DNE}\}$, be defined by $f(A)$ is the minimum element of A, if $A \neq \emptyset$ and $f(\emptyset) = \text{DNE}$.

Let $g : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{N})$ be the (restriction) function $g(A) = A \cap \mathbb{N}$.

(1) Compute $f(\{2, 3, 5\})$.

(2) Compute $f(\mathbb{N})$.

(3) Compute $f(\emptyset)$.

(4) Compute $f \circ g([1.5, 4) \cup [4.5, 5.5])$.

(5) Compute $f \circ g(\mathbb{R})$.

(6) Compute $f \circ g([-2, 0])$.

(7) In words, explain what the function $f \circ g$ does.

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(4) $\mathbb{Z} \cup \{x + \frac{1}{2} : x \in \mathbb{Z}\}$

(5) $\{1, 10, 100, 1000, \ldots, \}$

(6) $\mathbb{N} \times \mathbb{Q}$

Activity 3. Identify which of the following sets are finite, countable, or uncountable. Give a short explanation for your choice (a complete proof is not necessary).

(1) $\{1, 2, 3, \ldots, 2019\}$

(2) $\mathbb{Z}$

(3) $A \cup \mathbb{R}$, where A is any set

(4) $\mathcal{P}(\mathcal{P}(\mathcal{P}(\{0, 1\})))$

(5) $\mathcal{P}(\mathbb{N}) \cap \mathbb{N}$

(6) $\mathbb{Q} \cap [0, 1]$

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