MAT135H5 F - FALL 2020 - WRITTEN ASSIGNMENT 1 - SOLUTIONS

Submission

- **You must submit your completed Written Assignment on Crowdmark by 6:00pm (EDT) Friday September 25, 2020**. You will be emailed a link from Crowdmark with information on how to submit your solutions.
- Late assignments (even by a couple seconds) will not be accepted.
- Consider submitting your assignment well before the deadline. (1400 students all trying to submit at 5:59pm will break Crowdmark.)
- You do not need to print out this assignment; you may submit clear pictures/scans of your work on lined paper, or screenshots of your work.
- You do not need to submit the cover page, the grading scheme, or Part 3 of Question 3.
- You must correctly orient/rotate and order your submission.
- If you require additional space, please insert extra pages.

What if I don't have a UTorid yet? How do I submit my solutions?

We will synchronize your Quercus information to Crowdmark frequently. When you receive your utoronto email, then you be emailed a link from Crowdmark with instructions on how to upload your solutions. If you do not have your utoronto email address by Thursday September 24, then please email the course coordinator (Mike Pawliuk) then.

Additional Instructions

You must justify and support your solution to each question. You should use full sentences.

Academic Integrity

You are encouraged to work with your fellow students while working on questions from the written assignments. However, the writing of your assignment must be done without any assistance whatsoever. Do not post partial or complete solutions to Piazza.

I affirm that this assignment represents entirely my own efforts. I confirm that:

- I have not copied any portion of this work.
- I have not allowed someone else in the course to copy this work.
- This is the final version of my assignment and not a draft.
- I understand the consequences of violating the University's academic integrity policies as outlined in the *Code of Behaviour on Academic Matters*.

By submitting solutions for grading I agree that the statements above are true. If I do not agree with the statements above, I will not submit my assignment and will consult the course coordinator (Mike Pawliuk) immediately.

Grading Scheme

This is the grading scheme that TAs will use when grading this assignment. You do not need to submit this page.

Question 1. [6 points]. Each part is worth 2 points: 1 point for a correct answer, and 1 point for a clear and complete solution with full sentences.
For this question only, one total point may be deducted if the pages are not oriented or ordered correctly.

Question 2 [3 points]. Each part is worth 1 points: 1 point for a correct answer with explanation. Award no points for correct answers if there is only algebra and no further explanation.

Question 3 [6 points]. Part 1 and Part 2 is worth 3 points: 1 point for a good start (possibly including a correctly labelled diagram), 1 point for a correct answer, and 1 point for a clear and complete explanation with full sentences. Part 3 should not be graded.

Academic integrity instructions for TAs

I have sent you a list of screenshots from various online sources where the solutions were posted before the deadline. Please be on the look out for these and for any similarities between different submissions.

If you suspect that you see an academic integrity issue you must do the following:

- Use Crowdmark's "tag" feature tag the solutions as both "AI" and "questionnumber_source_version". For example, it might be "Q1_GC1_1". This will help me sort through things later, the tags are invisible to students.
- If you find suspiciously solutions, but they are not similar to an online source, then please tag them as "AI" and "questionnumber_groupN" (where N changes based on the number of similar groups you find).
- Please record your comments and observations and forward them to the course coordinator.
- You only need to tag and forward cases you believe are likely AI.

Once the grading is complete, an instructor will review your comments, and then set up a meeting with the student. Instructors and TAs cannot assign sanctions; that must happen at the department level.

Question 1. For this question, let $f : [0, +\infty) \to \mathbb{R}$ be defined by $f(x) = \sqrt{x}$, and let $f^{-1}(x)$ be its inverse function.

(1) What is the domain and range of the function $f^{-1}(x)$? Justify your answer.

Solution. The domain of f is given as $[0, +\infty)$, therefore the range of f^{-1} is $[0, +\infty)$. Note that the range of f is actually $[0, +\infty)$, therefore the domain of f^{-1} is $[0, +\infty)$.

Comment. Yes, even though we normally think of x^2 as having full domain $\mathbb{R}$, in the context of being the inverse of $\sqrt{x}$ we must restrict its domain to only $[0, +\infty)$. Also, note that this question did <u>not</u> ask you to compute f^{-1}, only its domain and range. Both of those things can be computed directly from $\sqrt{x}$.

(2) What are the domains and ranges of the functions $f \circ f^{-1}(x)$ and $f^{-1} \circ f(x)$? Justify your answer.

Solution. Since both f and f^{-1} have the same domain and ranges, namely $[0, +\infty)$, both compositions must have domain and range equal to $[0, +\infty)$.

(3) Are $(\sqrt{x})^2$ and $\sqrt{x^2}$ always equal for all real numbers x? Justify your answer.

Solution. No, since they are not the same for $x = -1$, since $\sqrt{x^2} = \sqrt{(-1)^2} = \sqrt{1} = 1$, but $(\sqrt{-1})^2$ is undefined.

Comment. To show that two expressions are not equal for all x, means you only need to give one example of an x where they are not equal. It would also be acceptable to point out that these two expression have different domains, and so are automatically not always equal.

This question was included because many students incorrectly claim "$\sqrt{x^2} = x$ for all real numbers x", when in fact this is only true for $x \geq 0$.

Question 2. According to the simple English Wikipedia page for Decibel:

> "A decibel (or dB) measures ratios of power or intensity. It expresses them as an exponential function. An increase of three decibels is approximately a doubling of power. Decibels are often used in measuring telecommunication signals."

According to the same article, calm breathing is approximately 10 dB, standing next to a jet engine is 150 dB, and 0 dB is the lowest amount of sound that (any) human can hear.

Comment. The intended solution requires only the information given in the statement of the question. This question was designed to assess a student's ability to extract a model from information given informally and support their choice of model.
The alternate solution we present here uses base 10, which students may have found by doing outside research or from previous knowledge. This accounts for the difference in solutions between the two methods. Indeed this is a more accurate model of decibels used by scientists. Even though this solution is not intended, it is of course, still acceptable.

(1) Describe the power $P(x)$ of a sound as a function of decibels x. Use the positive constant E for the amount of power for a sound with 0 dB.

Solution. (Intended) I claim that $P(x) = E2^{\frac{x}{3}}$.
Note that $P(0) = E2^{\frac{0}{3}} = E$, so we know that for 0 dB we get a power of E.
Also $P(x+3) = E2^{\frac{(x+3)}{3}} = E2^{\frac{x}{3}+1} = E2^{\frac{x}{3}}2 = 2P(x)$, so we know that "an increase of three decibels is exactly a doubling of power."

Solution. (Alternate) I claim that $P(x) = E10^{\frac{x}{10}}$.
Note that $P(0) = E2^{\frac{0}{10}} = E$, so we know that for 0 dB we get a power of E.
Also $P(x+3) = E10^{\frac{(x+3)}{10}} = E10^{\frac{x}{10}+\frac{3}{10}} = E2^{\frac{x}{3}}10^{\frac{3}{10}} \approx 1.995P(x)$, so we know that "an increase of three decibels is approximately a doubling of power."

(2) Justify why $P(x)$ is an exponential function. (What is its base? How does E affect it?)

Solution. (Intended) It is exponential because its base is 2, and $E \neq 0$.

Solution. (Alternate) It is exponential because its base is 10, and $E \neq 0$.

Comment. If $E = 0$ then this would be the constant 0 function, which (in my opinion) should not be considered an exponential function.

(3) Approximately how much more power is released by the sound of a jet engine compared to calm breathing? Give your answer as a multiplying factor. Explain your answer.

Solution. (Intended) From the question, the power released by calm breathing is $P(10)$ and for a jet engine it is $P(150)$.
We need to find the constant C so that $CP(10) = P(150)$. In other words

$$C = \frac{P(150)}{P(10)} = \frac{E2^{150/3}}{E2^{10/3}} = 2^{150/3-10/3} = 2^{140/3} \approx 1.12 \times 10^{14} = 112 \text{ quadrillion.}$$

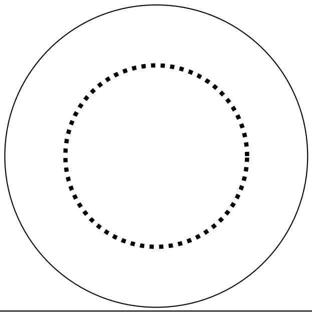
Solution. (Alternate) From the question, the power released by calm breathing is $P(10)$ and for a jet engine it is $P(150)$.
We need to find the constant C so that $CP(10) = P(150)$. In other words

$$C = \frac{P(150)}{P(10)} = \frac{E10^{150/10}}{E10^{10/10}} = 10^{150/10-10/10} = 10^{14} = 100 \text{ quadrillion.}$$

Question 3. Alice and Bob are pizza anarchists, which means they enjoy cutting their pizza into two parts of equal area in unusual ways. They have a circular pizza of positive radius R.

(1) For their first experiment, they cut a circular disk out of a pizza. Alice gets the smaller disk, and Bob gets the outer ring of pizza.

Find the radius r of this smaller disk that will make Alice and Bob have equal amounts of pizza. Justify your solution.



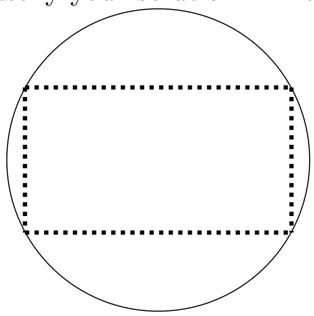
Solution. Let r, R be defined as in the statement of the problem. The area of the larger pizza is πR^2 and the area of the smaller disk is πr^2. Since we want the smaller disk to be half of the area of the larger pizza, we want

$$\pi r^2 = \frac{\pi R^2}{2}.$$

Solving for r gives $r^2 = \dfrac{R^2}{2}$ and so $r = \dfrac{R}{\sqrt{2}}$.

(2) For their second experiment, they cut a rectangle out of a pizza so that all four of its corners touch the edge of the pizza. Alice gets the rectangular piece, and Bob gets the 4 other pieces.

Find the dimensions x, y of this rectangle that will make Alice and Bob have equal amounts of pizza. Justify your solution. Hint: What is the diagonal?



Solution. (Intended solution) Let x, y, and R be as defined in the statement of the problem. The area of the circular pizza is again πR^2, and the area of the rectangle is xy. We want

$$xy = \frac{\pi R^2}{2}, \qquad \text{or equivalently} y = \frac{\pi R^2}{2x}.$$

Note that the diagonal of the rectangle is also the diameter of the circle. By Pythagoras, the length of this diagonal is given by $(2R)^2 = x^2 + y^2$. Combining these, we simplify towards a quartic equation (degree 4):

$$\begin{aligned}
& x^2 + y^2 = 4R^2 \\
\implies & x^2 + \left(\frac{\pi R^2}{2x}\right)^2 = 4R^2 && \text{(Substitute } y\text{)} \\
\implies & x^2 + \frac{\pi^2 R^4}{4x^2} = 4R^2 && \text{(Distribute the square)} \\
\implies & 4x^4 + \pi^2 R^4 = 16R^2x^2 && \text{(Multiply through by } 4x^2\text{)} \\
\implies & 4x^4 - 16R^2x^2 + \pi^2R^4 = 0
\end{aligned}$$

This is actually a quadratic in x^2 in disguise, so we can use the quadratic formula to solve for x^2. The quadratic formula gives us:

$$\begin{aligned}
x^2 &= \frac{-(-16R^2) \pm \sqrt{(-16R^2)^2 - 4(4)(\pi^2R^4)}}{2(4)} \\
&= \frac{16R^2 \pm \sqrt{(16^2R^4 - 16\pi^2R^4}}{8} \\
&= \frac{16R^2 \pm \sqrt{16R^4}\sqrt{16-\pi^2}}{8} \\
&= \frac{16R^2 \pm 4R^2\sqrt{16-\pi^2}}{8} \\
&= \frac{R^2}{2}\left(4 \pm \sqrt{16-\pi^2}\right)
\end{aligned}$$

Note that since $\sqrt{16-\pi^2} \approx 2.48$ both the positive and negative choices for $\pm$ will make $4 \pm \sqrt{16-\pi^2}$ positive.

Solving for x gives two solutions (since by the physical nature of the problem, we cannot have negative lengths of pizza):

$$x = \frac{R}{\sqrt{2}}\sqrt{\left(4 \pm \sqrt{16-\pi^2}\right)}$$

Then we solve for y by our first area formula:

$$y = \frac{\pi R^2}{2x} = \frac{\pi R^2}{2\frac{R}{\sqrt{2}}\sqrt{\left(4 \pm \sqrt{16-\pi^2}\right)}} = \frac{\pi R}{\sqrt{2}\sqrt{\left(4 \pm \sqrt{16-\pi^2}\right)}}.$$

So the pairs of dimensions are either both x and y choose the positive part for $\pm$, or they both choose the negative part.

Comment. Let x_+, y_+ be the psoitive choices for x, and let x_-, y_- be the negative choices. In this solution

$$\begin{aligned}
x_+ &\approx 1.799R \\
x_- &\approx 0.873R \\
y_+ &\approx 0.873R \\
y_- &\approx 1.799R
\end{aligned}$$

This suggests that the "two pairs of solutions" are really just one pair of solutions where one rectangle is long $\times$ short and one is turned 90 degrees to be short $\times$ long.

Comment. Checking that these solutions satisfy $xy = \dfrac{\pi R^2}{2}$ is straightforward, but checking that they satisfy $x^2 + y^2 = 4R^2$ appears to be challenging. (I, Mike, spent 2 hours trying to grind through it, and I kept getting stuck.)

Solution. (Solution that requires a clever idea) Let x, y, and R be as defined in the statement of the problem. The area of the circular pizza is again πR^2, and the area of the rectangle is xy. We want

$$xy = \frac{\pi R^2}{2}, \qquad \text{or equivalently } 2xy = \pi R^2.$$

Note that the diagonal of the rectangle is also the diameter of the circle. By Pythagoras, the length of this diagonal is given by $(2R)^2 = x^2 + y^2$.
By adding these two equations, or by subtracting them, we get these two equations:

$$x^2 + 2xy + y^2 = 4R^2 + \pi R^2$$
$$x^2 - 2xy + y^2 = 4R^2 - \pi R^2$$

The left hand sides are both perfect squares, so we get

$$(x+y)^2 = 4R^2 + \pi R^2 \qquad \Longrightarrow x + y = \sqrt{4R^2 + \pi R^2} = R\sqrt{4+\pi}$$
$$(x-y)^2 = 4R^2 - \pi R^2 \qquad \Longrightarrow x - y = \sqrt{4R^2 - \pi R^2} = R\sqrt{4-\pi}$$

This gives us a simple set of linear equations (that we will encounter again in calculus). Adding them, or subtracting them gives:

$$2x = R\sqrt{4+\pi} + R\sqrt{4-\pi} \qquad \Longrightarrow x = \frac{R\sqrt{4+\pi} + R\sqrt{4-\pi}}{2}$$
$$2y = R\sqrt{4+\pi} - R\sqrt{4-\pi} \qquad \Longrightarrow y = \frac{R\sqrt{4+\pi} - R\sqrt{4-\pi}}{2}$$

Comment. This solution is inspired by two observations from high school:

$$x^2 + 2xy + y^2 = (x+y)^2$$
$$x^2 - 2xy + y^2 = (x-y)^2$$

and if A, B are constants, then the system:

$$x + y = A$$
$$x - y = B$$

has solutions:

$$x = \frac{A+B}{2}$$
$$y = \frac{A-B}{2}.$$

Comment. The approximate solutions are:

$$x \approx 1.799R$$
$$y_+ \approx 0.873R$$

which line up with the intended solution's result. (It is a great algebra exercise to try to derive the "clever" x from the "intended" x. I don't know how to do it. -Mike)

Comment. This question was intended to be challenging, and have you get stuck (possibly multiple times). Getting stuck is a very natural part of doing mathematics. We all get stuck, instructors included. However, **we are not defined by when we get stuck, we are defined by how we respond to it**.
Ask questions, read the textbook, try again, get stuck again, ask the same question on Piazza again, realize you don't understand how the quadratic formula works, watch videos, get it (!), fix it, succeed.
Most importantly, it was designed to have you eventually succeed using only tools from high school. We want you to succeed in this course. You got this!

Comment. There can be many different solutions to a math problem. Be on the look out for alternate solutions that involve looking at a situation from a different perspective, or involves putting things together in a different way.
In this problem there was the "grindy, bash-your-way-through-it-with-algebra" solution, and the relatively elegant solution that required an observation, or new perspective. This is one of the things that makes math beautiful and fun.
Even if you didn't see the clever solution, <u>now</u> you know about it, and you can put it in your tool box for later. I promise we'll see the "$x + y = A, x - y = B$" system again in MAT136.

(3) **This part is not for marks, and you should not submit it.** Alice and Bob have been trying to evenly split up the pizza by giving Alice a triangular piece of pizza whose corners all touch the edge of the circular pizza.

Show that this is impossible for them to do no matter how they make the triangle.