MAT136H5 F - FALL 2023 - WRITTEN ASSIGNMENT 1 - SOLUTIONS

Submission

- **You must submit your completed Written Assignment on Crowdmark by 6:00pm (EDT) Friday September 22, 2023**. You will be emailed a link from Crowdmark with information on how to submit your solutions.
- **Late penalty**: Written assignments may be submitted up to 50 hours late at a penalty of 2% per hour late.
- Consider submitting your assignment well before the deadline.
- You do not need to print out this assignment; you may submit clear pictures/scans of your work on lined paper, or screenshots of your work.
- You do not need to submit the cover page, or the grading scheme.
- You must correctly orient/rotate and order your submission.
- If you require additional space, please insert extra pages.

Additional Instructions

You must justify and support your solution to each question. You should use full sentences.

Academic Integrity

You are encouraged to work with your fellow students while working on questions from the written assignments. However, the writing of your assignment must be done without any assistance whatsoever. Do not post partial or complete solutions to Piazza.

I affirm that this assignment represents entirely my own efforts. I confirm that:

- I have not copied any portion of this work.
- I have not allowed someone else in the course to copy this work.
- I have included the names of all my group members for Q1 (or I worked alone).
- This is the final version of my assignment and not a draft.
- I understand the consequences of violating the University's academic integrity policies as outlined in the *Code of Behaviour on Academic Matters*.

By submitting solutions for grading I agree that the statements above are true. If I do not agree with the statements above, I will not submit my assignment and will consult a course instructor (Mike Pawliuk) immediately.

Need help?

This problem set is designed to make you think, and it contains problems you've never seen before. We expect you'll need to come back to this assignment multiple times and try different approaches; we don't expect you to solve everything in one sitting. It's normal to get stuck! Every time you get stuck that means you're about to learn something when you get unstuck. Look for those moments!

If you're stuck for more than a day or two, you may want to ask for help. Here are some places to do that:

- **Ask on Piazza**. (If you want to post some of your work, please make it a private post.)
- **Office hours**. See Quercus for times and locations. There are 3 hours a week, and you can attend the office hours of any instructor or TA, not just the one for your LEC section.
- **Math Learning Center**. DH 2027.

Good luck, have fun!

Grading Scheme

This is the grading scheme that TAs will use when grading this assignment. You do not need to submit this page.

Question 1. [10 points].

(1) 1 point. All or nothing for a correct relationship.
(2) 1 point. 1 point for 4 or more properties correct, 0.5 for 2 or 3 of these properties correct. Do not award any points if there is little to no explanation, or the explanation is not at the appropriate (non-technical) level.
(3) 1 point. 0.5 points for each correct parameter.
(4) 1 point. 1 point for 4 or more properties correct, 0.5 for 2 or 3 of these properties correct.
(5) 1 point. 0.5 points for correct answer, 0.5 points for clear explanation.
(6) 1 point. 0.5 points for correct answer, 0.5 points for clear explanation.
(7) 1 point. 0.5 points for both correct answers, 0.5 points for clear explanation.
(8) 1 point. 0.5 points for correct answer, 0.5 points for clear explanation.
(9) 1 point. 0.5 points for correct answer, 0.5 points for clear explanation.
(10) 1 point. 0.5 for a correct answer, 0.5 for a reasonable justification with a source. (The source can be the article given in the assignment.)

Question 1. Garlic Mustard is a non-native, invasive plant that is established on the UTM campus. It is edible, and has a garlic taste. Read about the plant and its impact on the environment here:

`https://www.invadingspecies.com/invaders/plants/garlic-mustard-2/`

In this problem you will use sequences to model the spread of this plant, and propose policy to limit its spread.

Let a_0 be the amount of garlic mustard plants on UTM campus as of September 1, 2023. Let a_n be the amount of garlic mustard plants on UTM campus on September 1, $2023 + n$.

Next step! If after this assignment you're interested in translating this knowledge into action, then you may be interested in these nature stewardship events happening in Mississauga in September and October 2023.

`https://www.eventbrite.ca/e/invasive-species-stewardship-event-tickets-624779651757`

(1) Read the above article. According to it, how does a_{n+4} relate to a_n? Write this as a linear relationship.

Solution. The article says "Stands of garlic mustard can double in size every four years."
This says that a_{n+4} is double a_n. In other words

$$a_{n+4} = 2a_n$$

(2) Is the sequence $\{a_n\}$ convergent? Bounded? Monotone? Increasing? Decreasing? Explain briefly in a way that a non-technical audience can understand. List all assumptions you are making.

Solution. Using the simplest model, that assumes unlimited growth forever, we get:

- The sequence is unbounded and diverges (to ∞). This is because the amount of plants will keep growing forever; it is an invasive species.
- The sequence is increasing (hence monotone, and not decreasing), because we expect the plants to continue growing year over year.

The major problem with this model is that it is impossible for the amount of plants to continue growing forever. At some point the plants would run out of space to expand in UTM. This is called the "carrying capacity" of the environment.

Taking this into account gives us a more accurate model, and gives different answers to the questions:

- The sequence is bounded (above by the carrying capacity) and converges to the carrying capacity.
- The sequence is still increasing.

See here for an illustration of this: `https://www.desmos.com/calculator/bhclmulrsi`
Both models/assumptions are fine, it just depends on how we're using them. In the case of this assignment, we are going to work with the simpler model (assuming unlimited growth). The math will be easier, and (hopefully) will give us a similar answer compared to using the more sophisticated model.

(3) The sequence a_n can be modelled as a function $a2^{n/b}$ where a, b are real constants (called parameters). Find parameters a, b that make this model agree with the data about garlic mustard given above.

Solution. First, we use the model $a_n = a2^{n/b}$ at the value $n = 0$. This gives

$$a_0 = a2^{0/b} = a2^0 = a$$

So in other words, the parameter $a = a_0$. This might seem a bit strange, but remember that a_0 is just some quantity. We don't know exactly what it is (unless we go count some plants), but we can still use it to answer our questions. It turns out, surprisingly, we won't need to know much about a_0 in order to conclude some interesting results (in parts 8,9, and 10).

To get b we'll use the model together with the relationship we discovered in part 2

$$a_{n+4} = 2a_n$$

We may as well do this for $n = 0$, so

$$a_4 = 2a_0$$

and using the model at $n = 4$ gives

$$a_4 = a_0 2^{4/b}$$

so then

$$a_4 = 2a_0$$
$$\implies a_0 2^{4/b} = 2a_0$$
$$\implies 2^{4/b} = 2$$

Here, since both quantities have base 2, their powers must be equal (and the power on the right is actually just 1). So we get

$$4/b = 1 \implies 4 = b$$

So the model is $a = a_0, b = 4$ which gives

$$a_n = a_0 2^{n/4}$$

(4) Is the model (function) you've created in part (3), convergent? Bounded? Monotone? Increasing? Decreasing? Does this agree with the results you found in part (2)?

Solution. Since we know that $a_0 > 0$ (since there is some garlic mustard on campus) this sequence is an increasing exponential function. It is increasing (and so monotone), unbounded, and diverges to ∞.

This agrees with what we discovered in part 2 (the model with unlimited growth).

Bonus. If you're interested in the model that takes into account carrying capacity, it looks like this:

$$a_n = \frac{Ca2^{n/4}}{(C - a) + a2^{n/4}}$$

where C is the carrying capacity. You can check that:

- Plugging in $n = 0$ returns a_0 on the right.
- Taking $\lim_{n\to\infty} a_n$ gives C, the carrying capacity.

(5) Approximately how many years will it take for there to be 8 times as much garlic mustard on campus as in 2023?

Solution. One approach is to use relationship $a_{n+4} = 2a_n$.

Since $8 = 2^3$, this question is asking "How many years will it take for the quantity to double three times?". Each doubling takes 4 years, so this will take $3 \cdot 4 = 12$ years.

So in 2035 there will be 8 times as much garlic mustard as in 2023.

Solution. Alternatively, we can use the model $a_n = a_0 2^{n/4}$ and answer the question "Which value of n gives $a_n = 8a_0$?". Let's solve for this n.

$$\begin{aligned}
&a_n = 8a_0 \\
\implies &a_0 2^{n/4} = 8a_0 \\
\implies &2^{n/4} = 8 = 2^3 \\
\implies &n/4 = 3 \qquad \text{[Same base means they must have equal exponents]} \\
\implies &n = 12
\end{aligned}$$

So in 2035 there will be 8 times as much garlic mustard as in 2023.

(6) Approximately how many years will it take for there to be 10 times as much garlic mustard on campus as in 2023?

Solution. We can use the model $a_n = a_0 2^{n/4}$ and answer the question "Which value of n gives $a_n = 10a_0$?". Let's solve for this n.

$$\begin{aligned}
&a_n = 10a_0 \\
\implies &a_0 2^{n/4} = 10a_0 \\
\implies &2^{n/4} = 10 \\
\implies &n/4 \ln 2 = \ln 10 \qquad [\text{ since } \ln 2^{n/4} = n/4 \ln 2] \\
\implies &\frac{n}{4} = \frac{\ln 10}{\ln 2} \\
\implies &n = \frac{4 \ln 10}{\ln 2} \approx 13.29 \text{ years}
\end{aligned}$$

So in 2037 there will be 10 times as much garlic mustard as in 2023.

(7) Approximately how much garlic mustard will be on campus in 2031? What about in 2123?

Solution. One approach is to use relationship $a_{n+4} = 2a_n$.

After 4 years (2027), there will be $2a_0$ garlic mustard (one doubling). After another 4 years (2031), there will be $4a_0$ garlic mustard (a second doubling). In other words, in 2031 there will be 4 times as much garlic mustard as there was in 2023.

Similarly, 2123 is 100 years after 2023, which is $\frac{100}{4} = 25$ segments of 4 years. So there will be 25 doublings of the initial amount in 2023. In other words, in 2123 there will be $2^{25} \approx 34$ million times as much garlic mustard as there was in 2023.

Solution. We can use the model $a_n = a_0 2^{n/4}$.

Since 2031 is 8 years after 2023, we use a_8 and notice

$$a_8 = a_0 2^{8/4} = 4a_0$$

In other words, in 2031 there will be 4 times as much garlic mustard as there was in 2023.

Since 2123 is 100 years after 2023, we use a_{100} and notice

$$a_{100} = a_0 2^{100/4} = 2^{25} a_0$$

In other words, in 2031 there will be $2^{25} \approx 34$ million times as much garlic mustard as there was in 2023.

(8) Suppose that every year you are able to cull (remove) 10% of all the garlic mustard on campus. Is this enough to guarantee that the amount of garlic mustard does not increase?

Solution. First let's work through a couple early years, then we'll write down a general formula.

For 2024 ($n = 1$), we will first remove 10% of a_0, and then we will add how much it grows ($2^{1/4}$). The $2^{1/4}$ comes from the fact that four of these factors will combine to 2, which means that the plants are doubling every 4 years.

So $a_1 = (a_0 - 0.1a_0)2^{1/4} = 0.9a_0 2^{1/4}$.

For 2025 ($n = 2$), we will first remove 10% of a_1, and then we will add much it grows ($2^{1/4}$).
So $a_2 = (a_1 - 0.1a_1)2^{1/4} = 0.9a_1 2^{1/4} = (0.9)^2 a_0 2^{2/4}$.

We can continue on in this way, but eventually we will recognize the general formula that

$$a_n = (0.9)^n a_0 2^{n/4}$$

Now the question becomes "Is this sequence increasing or decreasing?". There are many ways to solve this (looking at Desmos, taking a derivative, etc.) that will give a correct solution. I'm going to show a different solution, using algebra.

$$\begin{aligned} a_n &= (0.9)^n a_0 2^{n/4} \\ &= (0.9)^{4n/4} a_0 2^{n/4} \\ &= \left((0.9)^4\right)^{n/4} a_0 2^{n/4} \\ &= a_0 \left(0.9^4 2\right)^{n/4} \end{aligned}$$

Now this looks a bit complicated, but really, it gives us a lot of information. In order to know if this sequence is increasing, we really need to know if the base of the exponent ($0.9^4 2$) is greater than 1 (so increasing), less than 1 (decreasing), or equal to 1 (constant).

Computing $0.9^4 2 \approx 1.3 > 1$, so this sequence is increasing.

In conclusion, culling 10% of the garlic mustard every year will not be enough to guarantee that the amount of plants does not increase.

(9) Suppose that every year you are able to cull 50% of all the garlic mustard on campus. Is this enough to guarantee that the amount of garlic mustard does not increase?

Solution. We did most of the work in part 8, so all we need to do is adapt the 10% to 50%. All that changes here is

$$a_n = (0.5)^n a_0 2^{n/4} = \ldots = a_0 \left(0.5^4 2\right)^{n/4}$$

Computing $0.5^4 2 \approx 0.1 < 1$, so this sequence is decreasing.

In conclusion, culling 50% of the garlic mustard every year will be enough to guarantee that the amount of plants does not increase.

(10) Briefly propose a policy to UTM that details the minimum percentage of garlic mustard that must be culled every year so that the amount of garlic mustard does not increase. Your policy proposal should include justification, with a source, for why it is important to limit the spread of Garlic Mustard on campus.

Solution. We did most of the work in part 8, so all we need to do is adapt the 10% to $p\%$, then compute which p will give a base of 1. All that changes here is

$$a_n = (1-p)^n a_0 2^{n/4} = \ldots = a_0 \left((1-p)^4 2\right)^{n/4}$$

We know $(1-p)^4 2 = 1$ when $(1-p)^4 = \frac{1}{2}$. So

$$1 - p = \frac{1}{2^{1/4}} \implies p = 1 - \frac{1}{2^{1/4}} \approx 0.1591 \approx 16\%$$

This is answer aligns with our results from part 8 (10% is not enough) and part 9 (50% is more than needed).

All that remains is to write our proposal to UTM. Your proposal will be different, and there are many ways to write such a thing. That's fine!

To the UTM administration,

Garlic Mustard is an invasive, non-native plant that is established on the UTM campus. It chokes out native plants, and is not a source of nutrition for wildlife. Many communities in Ontario are working to remove garlic mustard that has taken root. See [1] for more details.

Based on established growth estimates of the plant (it doubles every 4 years), and using mathematical models, I estimate that UTM will need to remove about 20% of the garlic mustard on campus every year in order to limit its growth. Any less than that (including doing nothing at all) will ensure that the invasive plant continues to spread on campus.

Thank you for your time, and I'm happy to explain my work in further detail.

Professor Mike Pawliuk
Department of Mathematical and Computational Science

Sources:

[1] `https://www.invadingspecies.com/invaders/plants/garlic-mustard-2/`